Representation Theory Lecture 6 07.08.2015

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Linear algebra Theorem 1:

$$\begin{pmatrix} 3 & 2 & 4 \\ 1 & 0 & 9 \\ -2 & -3 & -4 \end{pmatrix} \xrightarrow{x_{12}(-3)} \begin{pmatrix} 0 & 2 & -23 \\ 1 & 0 & 9 \\ -2 & -3 & -4 \end{pmatrix} \xrightarrow{x_{23}(\frac{1}{2})} \begin{pmatrix} 0 & 2 & -23 \\ 0 & -\frac{3}{2} & 7 \\ -2 & -3 & -4 \end{pmatrix}$$

$$\xrightarrow{x_{12}(\frac{2}{3}\cdot 2)} \begin{pmatrix} 0 & 0 & \frac{28}{3}-23 \\ 0 & -\frac{3}{2} & 7 \\ -2 & -3 & -4 \end{pmatrix} \xrightarrow{w_0} \begin{pmatrix} -2 & -3 & -4 \\ 0 & -\frac{3}{2} & 7 \\ 0 & 0 & -\frac{41}{3} \end{pmatrix} \xrightarrow{h_1(\frac{-1}{2})} \begin{pmatrix} 1 & \frac{3}{2} & 2 \\ 0 & -\frac{3}{2} & 7 \\ 0 & 0 & -\frac{41}{3} \end{pmatrix}$$

$$\xrightarrow{h_2(\frac{-2}{3})} \begin{pmatrix} 1 & \frac{3}{2} & 2 \\ 0 & 1 & -\frac{14}{3} \\ 0 & 0 & -\frac{41}{3} \end{pmatrix} \xrightarrow{h_3(\frac{-3}{41})} \begin{pmatrix} 1 & \frac{3}{2} & 2 \\ 0 & 1 & -\frac{14}{3} \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{x_{12}(\frac{-3}{2})} \begin{pmatrix} 1 & 0 & 9 \\ 0 & 1 & -\frac{14}{3} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{x_{13}(-9)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{14}{3} \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{x_{23}(\frac{+14}{3})} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = x_{23}\left(\frac{14}{3}\right) x_{13}(-9)\, x_{12}\left(\frac{-3}{2}\right) h_3\left(\frac{-3}{41}\right) h_2\left(\frac{-2}{3}\right) h_1\left(\frac{-1}{2}\right) w_0\, x_{12}\left(\frac{4}{3}\right) x_{23}\left(\frac{1}{2}\right) x_{12}(-3) \begin{pmatrix} 3 & 2 & 4 \\ 1 & 0 & 9 \\ -2 & -3 & -4 \end{pmatrix}$$

where

$$x_{12}(c) = \begin{pmatrix} 1 & c & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} \quad x_{13}(c) = \begin{pmatrix} 1 & 0 & c \\ & 1 & 0 \\ & & 1 \end{pmatrix} \quad x_{23} = \begin{pmatrix} 1 & 0 & 0 \\ & 1 & c \\ & & 1 \end{pmatrix}$$

$$w_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad h_1(c) = \begin{pmatrix} c & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad h_2(c) = \begin{pmatrix} 1 & & \\ & c & \\ & & 1 \end{pmatrix} \quad h_3(c) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & c \end{pmatrix}$$

Theorem The group $GL_n(\mathbb{C})$ is presented by generators

$$x_{ij}(c), \text{ for } c \in \mathbb{C} \text{ and } i,j \in \{1,\dots,n\} \text{ with } i \neq j,$$
$$h_i(c), \text{ for } c \in \mathbb{C}^\times \text{ and } i \in \{1,\dots,n\},$$
$$w \in S_n$$

with relations

$$S_n \text{ is a subgroup of } GL_n(\mathbb{C}),$$
$$h_i(c_1)h_i(c_2) = h_i(c_1c_2) \quad \text{and} \quad h_i(c_1)h_j(c_2) = h_j(c_2)h_i(c_1)$$
$$wh_i(c)w^{-1} = h_{w(i)}(c) \quad \text{and} \quad wx_{ij}(c)w^{-1} = x_{w(i),w(j)}(c)$$

and

$$x_{ij}(c_1)x_{ij}(c_2) = x_{ij}(c_1+c_2)$$
$$x_{ij}(c_1)x_{k\ell}(c_2) = x_{k\ell}(c_2)x_{ij}(c_1), \quad \text{if } i \neq \ell \text{ and } j \neq k$$
$$x_{ij}(c_1)x_{j\ell}(c_2) = x_{j\ell}(c_2)x_{ij}(c_1)x_{i\ell}(c_1c_2) \quad \text{if } i \neq \ell.$$
$$x_{ij}(c_1)x_{ki}(c_2) = x_{ki}(c_2)x_{ij}(c_1)x_{kj}(-c_1c_2) \quad \text{if } j \neq k$$

and

$$x_{ij}(c)x_{ji}(-c^{-1})x_{ij}(c) = h_j(-c)h_i(c)s_{ij}.$$

where

$$s_{ij} = \begin{array}{c} \\ i \\ j \end{array}\!\!\begin{array}{c} \begin{array}{ccccccc} & & i & & j & & \end{array} \\ \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & \ddots & & & \\ & & 1 & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \end{array}$$

$GL_n(\mathbb{C})$ is a complex algebraic group.

Our favorite algebraic group is $\mathbb{C}^\times = GL_1(\mathbb{C})$.

Wouldn't it be nice if $GL_n(\mathbb{C})$ was generated by subgroups isomorphic to $\mathbb{C}^\times$?

~~$\mathfrak{h}_{\mathbb{Z}} = \mathrm{Hom}(\mathbb{C}^\times, GL_n(\mathbb{C}))$~~

Let T be a maximal subgroup isomorphic to $\mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$.

$$T = \left\{ \begin{pmatrix} c_1 & & & 0 \\ & c_2 & & \\ & & \ddots & \\ 0 & & & c_n \end{pmatrix} \right\} \subseteq GL_n(\mathbb{C})$$

Note: T is the subgroup of $GL_n(\mathbb{C})$ generated by $h_1(c_1), \ldots, h_n(c_n)$ for $c_1, c_2, \ldots, c_n \in \mathbb{C}^\times$.

Then the normalizer of T in $GL_n(\mathbb{C})$ is

$$N = \left\{ \begin{matrix} n\times n \text{ with exactly one nonzero entry} \\ \text{in each row and each column} \end{matrix} \right\}$$

$$= \left\{ \begin{pmatrix} 0 & 0 & c_1 & 0 \\ c_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_3 \\ 0 & c_4 & 0 & 0 \end{pmatrix} \;\middle|\; c_1, \ldots, c_n \in \mathbb{C}^\times \right\}$$

$$= \left\{ \begin{pmatrix} c_1 & & & 0 \\ & c_2 & & \\ & & c_3 & \\ 0 & & & c_4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \;\middle|\; c_1, c_2, \ldots, c_n \in \mathbb{C}^\times \right\}$$

$$= T \cdot S_n$$

So $S_n = N/T$.

Then the subgroups

$$U^+ = \langle x_{ij}(c) \mid 1 \le i < j \le n, c \in \mathbb{C} \rangle$$

$$= \left\{ \begin{pmatrix} 1 & c_{12} & c_{13} & c_{14} \\ 0 & 1 & c_{23} & c_{24} \\ 0 & 0 & 1 & c_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} = \left\{ \begin{matrix} n \times n \text{ unipotent} \\ \text{upper triangular} \\ \text{matrices} \end{matrix} \right\}$$

$$U^- = \langle x_{ji}(c) \mid 1 \le j < i \le n, c \in \mathbb{C} \rangle$$

$$= \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ c_{21} & 1 & 0 & 0 \\ c_{31} & c_{32} & 1 & 0 \\ c_{41} & c_{42} & c_{43} & 1 \end{pmatrix} \right\} = \left\{ \begin{matrix} n \times n \text{ unipotent} \\ \text{lower triangular} \\ \text{matrices} \end{matrix} \right\}$$

$$B = \left\langle x_{ij}(c), h_\ell(d) \;\middle|\; \begin{matrix} i, j, \ell \in \{1, \ldots, n\} \\ i < j, c \in \mathbb{C}, d \in \mathbb{C}^\times \end{matrix} \right\rangle$$

$$= \left\{ \begin{matrix} n \times n \text{ upper triangular} \\ \text{matrices} \end{matrix} \right\} = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$$

and $B = TU^+$ and $U^+ = [B, B]$

where

$$[B, B] = \{ [b_1, b_2] \mid b_1, b_2 \in B \}$$

with $[x, y] = xyx^{-1}y^{-1}$ (the commutator of x and y).

WARNING: This bracket is <u>not</u> a Lie algebra bracket.