Representation Theory Lecture 4, 04.08.2015

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Let A be an $\mathbb{F}$-algebra and M an A-module.

A <u>composition series</u> of M is a sequence of submodules

$$M \supseteq M_1 \supseteq M_2 \supseteq \cdots$$

such that M_i/M_{i-1} is a simple A-module.

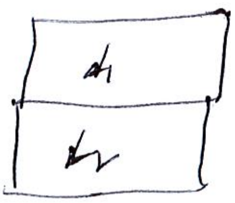
Presentations

The <u>Temperley-Lieb algebra</u> is

$$TL_k = \mathbb{C}\text{-span}\left\{ \begin{array}{c} \text{diagrams on } k \text{ dots} \\ \text{with no crossings} \end{array} \right\}$$

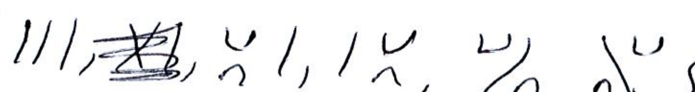
with product

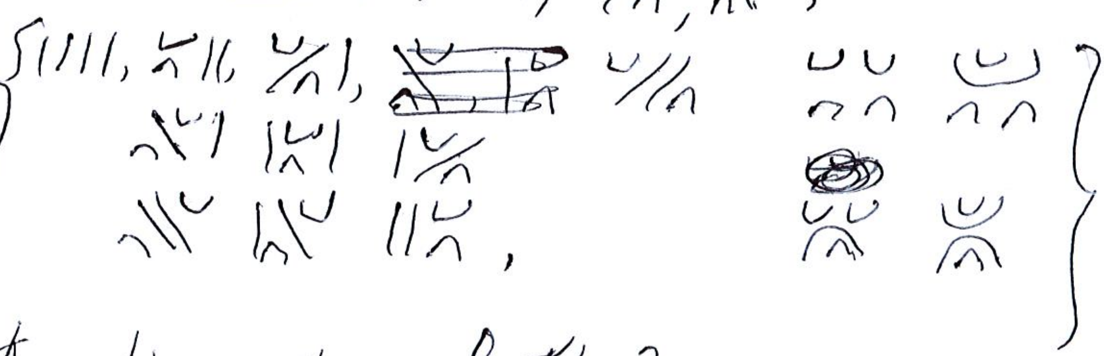
$$d_1 d_2 = (q+q^{-1})^{\#\text{internal loops}}$$



$TL_1 = \mathbb{C}\text{-span}\{ | \}$

$TL_2 = \mathbb{C}\text{-span}\{$  $\}$

$TL_3 = \mathbb{C}\text{-span}\{$  $\}$

$TL_4 = \mathbb{C}\text{-span}\{$  $\}$

HW: What is the dimension of TL_k?

<u>Theorem</u> The Temperley-Lieb algebra TL_k is presented by generators $e_1, e_2, \ldots, e_{k-1}$ and relations

$$e_i^2 = (q+q^{-1})e_i, \quad e_i e_{i\pm1} e_i = e_i \quad \text{and} \quad e_i e_j = e_j e_i \ \text{if } j \neq i\pm1.$$

<u>Proof</u> To show:

(1) Generators B can be written in terms of Generators A.

(2) Relations B can be derived from Relations A

(3) Generators A can be written in terms of Generators B.

(4) Relations A can be ~~writt~~ derived from Relations B

(1) $e_i = ||\cdots|\overset{i\ i+1}{\substack{\cup\\ \cap}}|\cdots|$

(2) $e_i^2 = \begin{matrix} ||\cdots|\substack{\cup\\ \cap}|\cdots| \\ ||\cdots|\substack{\cup\\ \cap}|\cdots| \end{matrix} = (q+q^{-1})\, ||\cdots|\substack{\cup\\ \cap}|\cdots| = (q+q^{-1})e_i$

$e_i e_{i+1} e_i = \begin{matrix} |\cdots|\substack{\cup\\ \cap}||\cdots| \\ |\cdots||\substack{\cup\\ \cap}|\cdots| \\ |\cdots|\substack{\cup\\ \cap}||\cdots| \end{matrix} = |\cdots|\overset{i\ i+1}{\substack{\cup\\ \cap}}|\cdots| = e_i.$

$e_i e_{i-1} e_i = \begin{matrix} |\cdots||\overset{i\ i+1}{\substack{\cup\\ \cap}}|\cdots| \\ |\cdots|\substack{\cup\\ \cap}||\cdots| \\ |\cdots||\substack{\cup\\ \cap}|\cdots| \end{matrix} = |\cdots||\overset{i\ i+1}{\substack{\cup\\ \cap}}|\cdots| = e_i$

$e_i e_j = \begin{matrix} |\cdots|\substack{\cup\\ \cap}|\cdots||||\cdots| \\ |\cdots||||\cdots|\substack{\cup\\ \cap}|\cdots| \end{matrix} = |\cdots|\substack{\cup\\ \cap}|\cdots|\substack{\cup\\ \cap}|\cdots| = \begin{matrix} |\cdots||||\cdots|\substack{\cup\\ \cap}|\cdots| \\ |\cdots|\substack{\cup\\ \cap}|\cdots||||\cdots| \end{matrix} = e_j e_i$

if $j \neq i\pm1$.

3 Let d be a diagram on k dots.

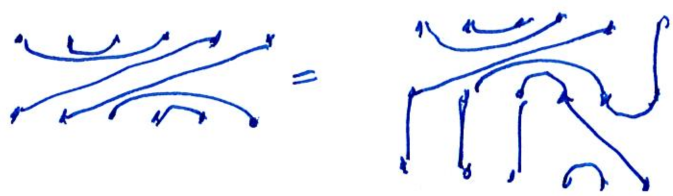
<u>Case 1</u>: $d = \boxed{d'}\ \big|_{k}^{k}$

<u>Case 2</u>: Let j be the index of the rightmost edge $\underset{j\ \ j+1}{\frown}$ in the bottom row. Then

$$d = d' e_{k-1} \cdots e_j \quad \text{with}$$

d' is d with the edge $\underset{j\ \ j+1}{\frown}$ removed and vertices $\underset{j+2}{\cdot}\ \underset{j+3}{\cdot} \cdots \underset{k}{\cdot}$ shifted to positions $\underset{j}{\cdot}\ \underset{j+1}{\cdot} \cdots \underset{k-1}{\cdot}$ and then vertical edge $\big|_{k}^{k}$.

For example:



Since d' is a diagram on $k-1$ dots, by induction, it can be written in the form

$$d' = e_{i_1} e_{i_1-1} \cdots e_{j_1} e_{i_2} e_{i_2-1} \cdots e_{j_2} \cdots e_{i_\ell} e_{i_\ell-1} \cdots e_{j_\ell}.$$

with $i_1 < i_2 \cdots < i_\ell$ and $j_1 < i_1,\ j_2 < i_2, \ldots$

4 $e_{k-1} e_{k-2} \cdots e_j\, e_r e_{r-1} \cdots e_\ell$

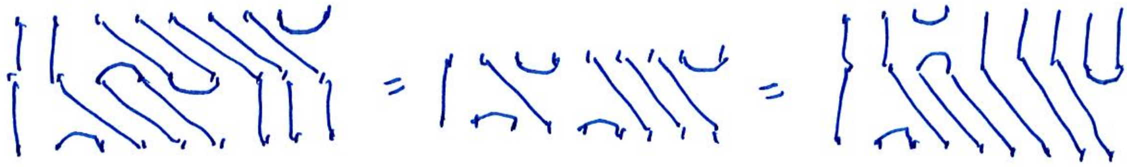
$= e_{k-1} e_{k-2} \cdots e_r e_{r-1} e_r e_{r-2} \cdots e_j\, e_{r-1} e_{r-2} \cdots e_\ell.$

$= e_{k-1} e_{k-2} \cdots e_{r+1} e_r\, e_{r-2} \cdots e_j\, e_{r-1} e_{r-2} \cdots e_\ell.$

$= e_{r-2} \cdots e_j\, e_{k-1} e_{k-2} \cdots e_r e_{r-1} \cdots e_\ell$

$= e_{r-2} \cdots e_j\, e_{k-1} \cdots e_\ell.$

For example: $(e_8 e_7 \cdots e_3)(e_5 e_4 e_3 e_2) = e_3(e_8 e_7 \cdots e_2)$



A <u>Lie algebra</u> is a vector space $\mathfrak{g}$ with a function $[,]: \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$ such that

(a) If $x, y \in \mathfrak{g}$ then $[y,x] = -[x,y]$

(b) If $x,y,z \in \mathfrak{g}$ then

$$[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0.$$

The Lie algebra $\mathfrak{sl}_2$ is

$$\mathfrak{sl}_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{matrix} a,b,c,d \in \mathbb{C} \\ a+d=0 \end{matrix} \right\}$$

with $[,]: \mathfrak{sl}_2 \otimes \mathfrak{sl}_2 \to \mathfrak{sl}_2$ defined by

$$[x,y] = xy - yx$$

(where the product on the right hand side is matrix multiplication).

<u>Proposition</u> The Lie algebra $\mathfrak{sl}_2$ is presented by generators e, f, h with relations

$$[e,f] = h, \quad [h,e] = 2e, \quad [h,f] = -2f.$$

The <u>enveloping algebra of $\mathfrak{sl}_2$</u> is the associative algebra generated by e, f, h with relations

$$ef - fe = h, \quad he - eh = 2e, \quad hf - fe = -2f.$$