What are the irreducible and indecomposable representations of $\mathbb{C}[t]$?

Theorem If A is commutative then all irreducible representations are 1-dimensional.

(Think about how this might be proved.
Dissertation Chapter I)

Any irreducible representation of $\mathbb{C}[t]$ is 1 dimensional:

For $\lambda \in \mathbb{C}$,
$$\begin{aligned} \rho_\lambda : \mathbb{C}[t] &\longrightarrow M_1(\mathbb{C}) \\ c &\longmapsto c, \quad \text{for } c \in \mathbb{C} \\ t &\longmapsto \lambda \end{aligned}$$

is an irreducible representation. As a $\mathbb{C}[t]$-module

$$M = \mathbb{C} v_\lambda \quad \text{with} \quad t v_\lambda = \lambda \cdot v_\lambda. \quad \text{(eigenvector)}$$

An eigenvector in V is an 1-dimensional $\mathbb{C}[t]$-submodule of V.

<u>d-dimensional representations of $\mathbb{C}[t]$</u>

$$\begin{aligned} \mathbb{C}[t] &\longrightarrow M_d(\mathbb{C}) \\ c &\longmapsto \begin{pmatrix} c & & 0 \\ & \ddots & \\ 0 & & c \end{pmatrix} \\ t &\longmapsto A = (a_{ij}) \end{aligned}$$

is the representation corresponding to

$$M = \operatorname{span}\{b_1, \dots, b_d\} \quad \text{with} \quad t b_i = \sum_{j=1}^{d} a_{ji} b_j$$

If we change basis on M by a change of basis matrix ~~P then~~ $P \in GL_d(\mathbb{C})$ then

$$\mathbb{C}[t] \longrightarrow M_d(\mathbb{C})$$

$$c \longmapsto \begin{pmatrix} c & & 0 \\ & \ddots & \\ 0 & & c \end{pmatrix} = P(cI_d)P^{-1}$$

$$t \longmapsto PAP^{-1}.$$

– Same module M.

So there is a one-to-one correspondence:

$$\left\{\begin{array}{c} d\text{-dimensional} \\ \text{representations} \\ \text{of } \mathbb{C}[t] \end{array}\right\} \longleftrightarrow \left\{\begin{array}{c} \text{Orbits of } GL_d(\mathbb{C}) \\ \text{acting on } M_d(\mathbb{C}) \\ \text{by conjugation} \end{array}\right\}$$

Another name for $M_d(\mathbb{C})$ is $\mathfrak{gl}_d(\mathbb{C})$, the Lie algebra of $GL_d(\mathbb{C})$.

Another name for the ~~adj~~ action of $GL_d(\mathbb{C})$ on $M_d(\mathbb{C})$ by conjugation is

<u>the adjoint action of $GL_d(\mathbb{C})$ on $\mathfrak{gl}_d(\mathbb{C})$</u>.

(Warning: There is also an "adjoint action of $\mathfrak{gl}_d(\mathbb{C})$ on $\mathfrak{gl}_d(\mathbb{C})$" which is a <u>different</u> thing)

The Jordan normal form theorem provides representatives of the orbits of $GL_d(\mathbb{C})$ acting on $\mathfrak{gl}_d(\mathbb{C})$ by conjugation.

As representations:

$$\mathbb{C}[t] \longrightarrow M_d(\mathbb{C})$$
$$t \longmapsto \left(\begin{array}{c|c} \begin{matrix}\lambda_1 & 1 \\ & \lambda_1\end{matrix} & 0 \\ \hline 0 & \begin{matrix}\lambda_2 & 1 & \\ & \lambda_2 & 1 \\ & & \lambda_2\end{matrix} \end{array}\right)$$

As ~~modules~~ $\mathbb{C}[t]$-modules

$$M = M_{\lambda_1,d_1} \oplus M_{\lambda_2,d_2} \oplus \cdots \oplus M_{\lambda_\ell,d_\ell} \quad \text{with } d_1+\cdots+d_\ell=d.$$

where M_{λ_i,d_i} corresponds to the representation

$$\mathbb{C}[t] \longrightarrow M_{d_i}(\mathbb{C})$$
$$t \longmapsto \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}$$

i.e. the module $M_i = \mathrm{span}\{v_1, v_2, \ldots v_{d_i}\}$

with $tv_1 = \lambda_i v_1$, $tv_2 = \lambda_i v_2 + v_1$, $\ldots$, $tv_d = \lambda_i v_d + v_{d-1}$.

and, if $a = t - \lambda_i$ then

$$av_1 = 0, \quad av_2 = v_1, \quad \ldots, \quad av_d = v_{d-1}.$$

Then let $V_1 = \mathrm{span}\{v_1\}$
$V_2 = \mathrm{span}\{v_1, v_2\}$
$V_3 = \mathrm{span}\{v_1, v_2, v_3\}$
$\vdots$

Then $0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{d-1} \subseteq V_d = M$

is a <u>composition series</u> of M with

$$\mathbb{C}_\lambda \cong V_j/V_{j-1} \text{ a } \underbrace{\text{1-dimensional}}_{\text{simple}} \ \mathbb{C}[t]\text{-module.}$$

<u>Hence:</u>

(a) The simple $\mathbb{C}[t]$-modules $\mathbb{C}_\lambda$ are indexed by $\lambda \in \mathbb{C}$,

$M = \mathbb{C}v_\lambda$ with $tv_\lambda = \lambda v_\lambda$.

(b) The indecomposable finite dimensional $\mathbb{C}[t]$-modules are indexed by pairs $\{(\lambda, d) \mid \lambda \in \mathbb{C}, d \in \mathbb{Z}_{>0}\}$

$M_{\lambda,d} = \operatorname{span}\{v_1, \ldots, v_d\}$ with

$tv_1 = \lambda v_1$, $tv_2 = \lambda v_2 + v_1, \ldots, tv_d = \lambda v_d + v_{d-1}$

(c) $M_{\lambda,d}$ has a composition series with d-factors all isomorphic to $M_{\lambda,1}$.

In the Grothendieck group:

$[M_{\lambda,d}] = d[M_{\lambda,1}]$.

The Grothendieck group of $\mathbb{C}[t]$ is

$\mathbb{Z}\text{-span}\{[M_{\lambda,1}] \mid \lambda \in \mathbb{C}\}$.