Let $M = \text{span}\{v_1, v_2\}$ with S_3 acting by

$$s_1 v_1 = v_1, \qquad s_2 v_1 = -\tfrac{1}{2} v_1 + \tfrac{3}{2} v_2$$
$$s_1 v_2 = -v_2, \qquad s_2 v_2 = \tfrac{1}{2} v_1 + \tfrac{1}{2} v_2$$

In matrix form

$$s_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}, \quad s_1 s_2 = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

$$s_2 s_1 = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \qquad s_1 s_2 s_1 = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix} = s_2 s_1 s_2$$

where the last equality follows from

$$s_2 \cdot s_1 s_2 = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} - \frac{3}{4} & -\frac{1}{4} - \frac{1}{4} \\ -\frac{3}{4} - \frac{3}{4} & \frac{3}{4} - \frac{1}{4} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}.$$

Then $M \otimes M$ has basis $\{v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_1, v_2 \otimes v_2\}$

and

$$s_1(v_1 \otimes v_1) = s_1 v_1 \otimes s_1 v_1 = v_1 \otimes v_1$$
$$s_1(v_1 \otimes v_2) = s_1 v_1 \otimes s_1 v_2 = -v_1 \otimes v_2$$
$$s_1(v_2 \otimes v_1) = -v_2 \otimes v_1$$
$$s_1(v_2 \otimes v_2) = v_2 \otimes v_2$$

so $s_1 = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$ in matrix form!

and
$$s_2(v_1 \otimes v_1) = \left(-\tfrac{1}{2} v_1 + \tfrac{3}{2} v_2\right) \otimes \left(-\tfrac{1}{2} v_1 + \tfrac{3}{2} v_2\right)$$
$$= \tfrac{1}{4}(v_1 \otimes v_1) - \tfrac{3}{4}(v_1 \otimes v_2) - \tfrac{3}{4}(v_2 \otimes v_1) + \tfrac{9}{4}(v_2 \otimes v_2)$$

$$s_2(v_2 \otimes v_2) = \left(\tfrac{1}{2} v_1 + \tfrac{1}{2} v_2\right) \otimes \left(\tfrac{1}{2} v_1 + \tfrac{1}{2} v_2\right)$$
$$= \tfrac{1}{4}(v_1 \otimes v_1) + \tfrac{1}{4}(v_1 \otimes v_2) + \tfrac{1}{4}(v_2 \otimes v_1) + \tfrac{1}{4}(v_2 \otimes v_2).$$

If $s_2(v_1 \otimes v_1 + \lambda(v_2 \otimes v_2)) = v_1 \otimes v_1 + \lambda(v_2 \otimes v_2)$ then

$\frac{1}{4}(1+\lambda)=1$ and $\frac{9}{4}+\lambda\frac{1}{4}=\lambda$

and $-\frac{3}{4}+\frac{\lambda}{4}=0$ and $-\frac{3}{4}+\frac{\lambda}{4}=0$.

So $\lambda=3$ and $\mathbb{C}(v_1\otimes v_1+3(v_2\otimes v_2))$ is a submodule of $M\otimes M$.

Furthermore this is the only copy of the trivial representation in $M\otimes M$.

Now

$$s_2(v_1\otimes v_2)=(-\tfrac{1}{2}v_1+\tfrac{3}{2}v_2)\otimes(\tfrac{1}{2}v_1+\tfrac{1}{2}v_2)$$

$$=-\tfrac{1}{4}(v_1\otimes v_1)-\tfrac{1}{4}(v_1\otimes v_2)+\tfrac{3}{4}(v_2\otimes v_1)+\tfrac{3}{4}(v_2\otimes v_2)$$

$$s_2(v_2\otimes v_1)=(\tfrac{1}{2}v_1+\tfrac{1}{2}v_2)\otimes(-\tfrac{1}{2}v_1+\tfrac{3}{2}v_2)$$

$$=-\tfrac{1}{4}(v_1\otimes v_1)+\tfrac{3}{4}(v_1\otimes v_2)-\tfrac{1}{4}(v_2\otimes v_1)+\tfrac{3}{4}(v_2\otimes v_2)$$

So if

$$s_2(v_1\otimes v_2+\lambda(v_2\otimes v_1))=-(v_1\otimes v_2+\lambda(v_2\otimes v_1))$$

then $-\frac{1}{4}-\frac{\lambda}{4}=0$, $-\frac{1}{4}+\frac{3}{4}\lambda=-1$, $\frac{3}{4}-\frac{\lambda}{4}=\lambda$, $\frac{3}{4}+\lambda\frac{3}{4}=0$.

So $\lambda=-1$ and $\mathbb{C}(v_1\otimes v_2-v_2\otimes v_1)$ is a submodule and is the only copy of the sign representation in M.

So $M\otimes M\simeq S^{\square\square\square}\oplus S^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}\oplus S^{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}$

(because $\dim(M\otimes M)=4$ and $\dim(S^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}})=2$).

$$s_1 s_2 s_1 (v_1 \otimes v_1) = \left(-\tfrac{1}{2} v_1 + -\tfrac{3}{2} v_2\right) \otimes \left(-\tfrac{1}{2} v_1 + -\tfrac{3}{2} v_2\right)$$

$$= \tfrac{1}{4}(v_1 \otimes v_1) + \tfrac{3}{4}(v_1 \otimes v_2) + \tfrac{3}{4}(v_2 \otimes v_1) + \tfrac{9}{4} v_2 \otimes v_2$$

$$s_1 s_2 s_1 (v_2 \otimes v_2) = \left(-\tfrac{1}{2} v_1 + \tfrac{1}{2} v_2\right) \otimes \left(-\tfrac{1}{2} v_1 + \tfrac{1}{2} v_2\right)$$

$$= \tfrac{1}{4}(v_1 \otimes v_1) - \tfrac{1}{4}(v_1 \otimes v_2) - \tfrac{1}{4} v_2 \otimes v_1 + \tfrac{1}{4} v_2 \otimes v_2$$

So

$$(s_2 + s_1 s_2 s_1)(v_1 \otimes v_1) = \tfrac{1}{2}(v_1 \otimes v_1) + \tfrac{18}{4}(v_2 \otimes v_2)$$

$$(s_2 + s_1 s_2 s_1)(v_2 \otimes v_2) = \tfrac{1}{2}(v_1 \otimes v_1) + \tfrac{1}{2}(v_2 \otimes v_2)$$

so, on span$\{v_1 \otimes v_1, v_2 \otimes v_2\}$,
$s_2 + s_1 s_2 s_1$ has matrix $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{18}{4} & \frac{1}{2} \end{pmatrix}$

and $v_1 \otimes v_1 + 3(v_2 \otimes v_2)$ has eigenvalue 2

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{9}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{4}{2} \\ \frac{12}{2} \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

and

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{18}{4} & \frac{1}{2} \end{pmatrix} - 2 = \begin{pmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{9}{2} & -\frac{3}{2} \end{pmatrix}$$

is a projection onto $\mathbb{C}\left(-\tfrac{3}{2}(v_1 \otimes v_1) + \tfrac{1}{2}(v_2 \otimes v_2)\right)$

which is $\mathbb{C}\left(-3(v_1 \otimes v_1) + v_2 \otimes v_2\right)$

$$s_1 s_2 s_1 (v_1 \otimes v_2) = \left(-\tfrac{1}{2} v_1 + -\tfrac{3}{2} v_2\right) \otimes \left(-\tfrac{1}{2} v_1 + \tfrac{1}{2} v_2\right)$$

$$= \tfrac{1}{4}(v_1 \otimes v_1) - \tfrac{1}{4} v_1 \otimes v_2 + \tfrac{3}{4} v_2 \otimes v_1 + -\tfrac{3}{4}(v_2 \otimes v_2)$$

$$s_1 s_2 s_1 (v_2 \otimes v_1) = \left(-\tfrac{1}{2} v_1 + \tfrac{1}{2} v_2\right) \otimes \left(-\tfrac{1}{2} v_1 + -\tfrac{3}{2} v_2\right)$$

$$= \tfrac{1}{4}(v_1 \otimes v_1) + \tfrac{3}{4}(v_1 \otimes v_2) - \tfrac{1}{4}(v_2 \otimes v_1) - \tfrac{3}{4}(v_2 \otimes v_2)$$

So

$$(s_2 + s_1 s_2 s_1)(v_1 \otimes v_2) = -\tfrac{1}{2}(v_1 \otimes v_2) + \tfrac{3}{2}(v_2 \otimes v_1)$$

$$(s_2 + s_1 s_2 s_1)(v_2 \otimes v_1) = \tfrac{3}{2}(v_1 \otimes v_2) - \tfrac{1}{2}(v_2 \otimes v_1).$$

So, on $\mathrm{span}\{v_1 \otimes v_2, v_2 \otimes v_1\}$

$s_2 + s_1 s_2 s_1$ has matrix $\begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$

and $v_1 \otimes v_2 - v_2 \otimes v_1$ has eigenvalue -2.

and $\begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} + 2 = \begin{pmatrix} \frac{3}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} \end{pmatrix}$ ~~has~~ is a projection

onto $\mathbb{C}(v_1 \otimes v_2 + v_2 \otimes v_1)$.

So
$$M \otimes M = \mathbb{C}(v_1 \otimes v_1 + 3(v_2 \otimes v_2))$$
$$\oplus\ \mathbb{C}\text{-span}\{-3(v_1 \otimes v_1) + v_2 \otimes v_2,\ v_1 \otimes v_2 + v_2 \otimes v_1\}$$
$$\oplus\ \mathbb{C}\text{-span}\{v_1 \otimes v_2 - v_2 \otimes v_1\}$$
as $\mathbb{C}S_3$-modules.