Representation Theory Lecture 28, 06 October 2015

$\mathfrak{g} = \mathfrak{sl}_3 = \{ x \in M_3(\mathbb{C}) \mid \operatorname{tr} x = 0 \}$

$\cup|$

$\mathfrak{b} = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\}$

$\cup|$

$\mathfrak{h} = \left\{ \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix} \;\middle|\; x_1 + x_2 + x_3 = 0 \right\}$

So that if $\varepsilon_i : \mathfrak{h} \to \mathbb{C}$, $\begin{pmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{pmatrix} \mapsto x_i$ then $\mathfrak{h}^* = \dfrac{\operatorname{span}\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}}{\langle \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0 \rangle}$.

Then

$$\mathfrak{g} = \Big(\bigoplus_{\alpha \in R^+} \mathfrak{g}_{-\alpha}\Big) \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha \quad \text{with}$$

$$R^+ = \{ \varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_3, \varepsilon_2 - \varepsilon_3 \} \quad \text{and} \quad \mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{C} E_{ij}.$$

and the simple roots are $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\varepsilon_2 - \varepsilon_3 = \alpha_2$.

If

$$e_1 = E_{12}, \quad e_2 = E_{23}, \quad f_1 = E_{21}, \quad f_2 = E_{32}$$

then $\mathfrak{sl}_3$ is generated by e_1, e_2, f_1, f_2 and relations

$$[e_1, f_1] = h_1 \qquad [h_1, e_1] = \alpha_1(h_1) e_1, \quad [h_1, e_2] = \alpha_2(h_1) e_2$$
$$[e_2, f_2] = h_2 \qquad [h_2, e_2] = \alpha_2(h_2) e_2, \quad [h_2, e_1] = \alpha_1(h_2) e_1$$
$$[e_1, [e_1, e_2]] = 0, \quad [e_2, [e_2, e_1]] = 0$$
$$[f_1, [f_1, f_2]] = 0, \quad [f_2, [f_2, f_1]] = 0$$

Let $\omega_1, \omega_2 \in \mathfrak{h}^*$ be given by

$$\omega_1(h_1) = 1, \quad \omega_2(h_1) = 0$$
$$\omega_1(h_2) = 0, \quad \omega_2(h_2) = 1.$$

Then $\mathfrak{h}^* = \operatorname{span}\{\omega_1, \omega_2\} = \operatorname{span}\{\alpha_1, \alpha_2\}$.

$\mathfrak{h}$ has basis $\{h_1, h_2\}$

$\mathfrak{n}^+$ has basis $\{e_1, e_2, e_{23}\}$ and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$

So $U = U^- U_0 U^+$ has basis

$$\{ f_1^a f_2^b f_{32}^c h_1^d h_2^e e_1^f e_2^g e_{23}^h \mid a,b,c,d,e,f,g,h \in \mathbb{Z}_{\geq 0} \}.$$

Let $\mu = \mu_1 \omega_1 + \mu_2 \omega_2 \in \mathfrak{h}^*$. Then

$$M(\mu) = U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}_\mu = \mathrm{span}\{ f_1^a f_2^b f_{32}^c v^+ \mid a,b,c \in \mathbb{Z}_{\geq 0}\}$$

with $e_1 v^+ = 0$, $e_2 v^+ = 0$, $e_{23} v^+ = 0$,

$h_1 v^+ = \mu_1 v^+$ and $h_2 v^+ = \mu_2 v^+$.

<u>Theorem</u> Let $\mu = \mu_1 \omega_1 + \mu_2 \omega_2 \in \mathfrak{h}^*$

(a) $M(\mu)$ has a unique simple head $L(\mu)$

$$\frac{M(\mu)}{\text{max. proper submodule}} = L(\mu)$$

(b) $M(\mu)$ is simple if and only if

$\mu_1 = \mu(h_1) \notin \mathbb{Z}_{\geq 0}$ and $\mu_2 = \mu(h_2) \notin \mathbb{Z}_{\geq 0}$.
and $\mu(h_1 + h_2) \notin \mathbb{Z}_{\geq 0}$.

(c) $L(\mu)$ is finite dimensional if and only if

$\mu_1 = \mu(h_1) \in \mathbb{Z}_{\geq 0}$ and $\mu_2 = \mu(h_2) \in \mathbb{Z}_{\geq 0}$.

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Composition series and characters

As an $\mathfrak{h}$-module

$$M(\mu) = \bigoplus_{\nu \in \mathfrak{h}^*} M(\mu)_\nu \quad \text{with}$$

$$M(\mu)_\nu = \{ m \in M(\mu) \mid \text{if } h \in \mathfrak{h} \text{ then } \nu(h)m = hm \}$$

The <u>character</u> of $M(\mu)$ is

$$\mathrm{char}(M(\mu)) = \sum_{\nu \in \mathfrak{h}^*} \dim(M(\mu)_\nu)\, e^{\nu}$$

Since $h(f_1^a f_2^b f_{32}^c v^+) = (\mu - a\alpha_1 - b\alpha_2 - c(\alpha_1+\alpha_2))(h) \cdot f_1^a f_2^b f_{32}^c v^+$

then
$$\mathrm{char}(M(\mu)) = \sum_{a,b,c \in \mathbb{Z}_{\geq 0}} e^{\mu - a\alpha_1 - b\alpha_2 - c(\alpha_1+\alpha_2)}$$

$$= e^{\mu} \frac{1}{1-e^{-\alpha_1}} \cdot \frac{1}{1-e^{-\alpha_2}} \cdot \frac{1}{1-e^{-(\alpha_1+\alpha_2)}}$$

$$= e^{\mu} \prod_{\alpha \in R^+} \frac{1}{1-e^{-\alpha}} = e^{\mu+\rho} \prod_{\alpha \in R^+} \frac{1}{e^{\alpha/2}-e^{-\alpha/2}}$$

<u>Theorem</u> If $\mu(h_1) \in \mathbb{Z}_{\geq 0}$ and $\mu(h_2) \in \mathbb{Z}_{\geq 0}$ then

$$\mathrm{char}(L(\mu)) = \left(\sum_{w \in S_3} \cancel{\det(w)}\, w \right) \frac{e^{\mu}}{\prod_{\alpha \in R^+}(1-e^{-\alpha})}$$