Representation Theory, Lecture 27, 25 September 2015 ①
Univ. of Melbourne

$U = U\mathfrak{sl}_2$ has basis $\{f^a h^b e^c \mid a,b,c \in \mathbb{Z}_{\geq 0}\}$

$U\mathfrak{b}$ has basis $\{h^b e^c \mid b,c \in \mathbb{Z}_{\geq 0}\}$.

Let $\mu \in \mathbb{C}$. Define a $U\mathfrak{b}$-module $\mathbb{C}_\mu = \mathbb{C}\text{-span}\{v^+\}$

with $hv^+ = \mu v^+$ and $ev^+ = 0$.

Then $h^b e^c v^+ = \begin{cases} \mu^b v^+, & \text{if } c = 0, \\ 0, & \text{if } c \neq 0. \end{cases}$

The Verma module of highest weight μ is

$$M(\mu) = U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}_\mu$$

If $A \subseteq B$ are algebras and M is an A-module then the induced module is

$$B \otimes_A M = \text{span}\{b \otimes m \mid b \in B, m \in M\}$$

with relations

$$ba \otimes m = b \otimes am, \quad \text{for } b \in B, a \in A, m \in M.$$

So $M(\mu) = \text{span}\{f^a h^b e^c \otimes v^+ \mid a,b,c \in \mathbb{Z}_{\geq 0}\}$

with $f^a h^b e^c \otimes v^+ = f^a \otimes h^b e^c v^+ = \begin{cases} \mu^b (f^a \otimes v^+), & \text{if } c = 0 \\ 0, & \text{if } c \neq 0. \end{cases}$

So $M(\mu) = \text{span}\{f^a v^+ \mid a \in \mathbb{Z}_{\geq 0}\}$.

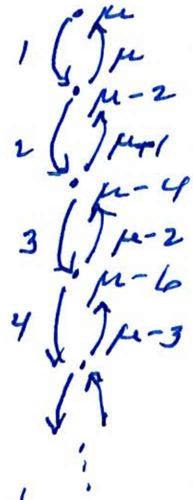
$= \text{span}\{f^{(a)} v^+ \mid a \in \mathbb{Z}_{\geq 0}\}$ where $f^{(a)} v^+ = \frac{1}{a!} f^a v^+$.

Prove, by induction, that

$$f\cdot f^{(a)}v^+ = (a+1)f^{(a+1)}v^+,$$

$$h\cdot f^{(a)}v^+ = (\mu-2a)f^{(a)}v^+,$$

$$e\cdot f^{(a)}v^+ = (\mu-(a-1))f^{(a-1)}v^+.$$



So, in the basis $\{f^{(a)}v^+ \mid a\in\mathbb{Z}_{\geq 0}\}$ the action of $U\mathfrak{sl}_2$ on $M(\mu)$ is given by the matrices

$$f = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 2 & 0 & & \\ & & 3 & 0 & \\ & & & \ddots & \ddots \end{pmatrix} \qquad e = \begin{pmatrix} 0 & \mu & & & \\ & 0 & \mu-1 & & \\ & & 0 & \mu-2 & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}$$

$$h = \begin{pmatrix} \mu & & & & \\ & \mu-2 & & & \\ & & \mu-4 & & \\ & & & \mu-6 & \\ & & & & \ddots \end{pmatrix}$$

Show that $M(\mu)$ is a simple $U\mathfrak{sl}_2$-module if and only if

$$0 \notin \{\mu-k \mid k\in\mathbb{Z}_{\geq 0}\}.$$

i.e. $M(\mu)$ is simple if and only if $\mu\notin\mathbb{Z}_{\geq 0}$.

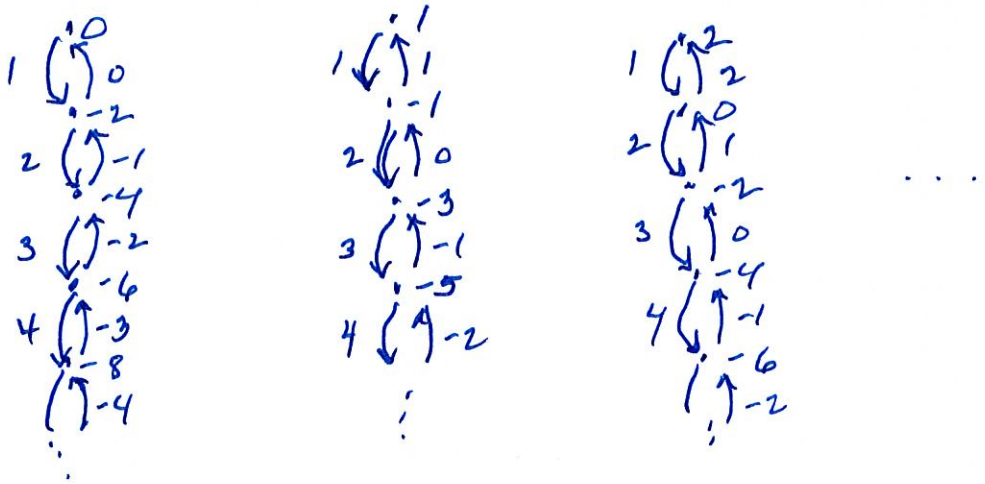
If $k\in\mathbb{Z}_{\ge 0}$ and $\mu=k$ then

$N=\mathrm{span}\{f^{(a)}v^+ \mid a\in\mathbb{Z}_{\ge k}\}=\mathrm{span}\{f^{(k)}v^+, f^{(k+1)}v^+,\dots\}$

is a submodule of $M(\mu)$.

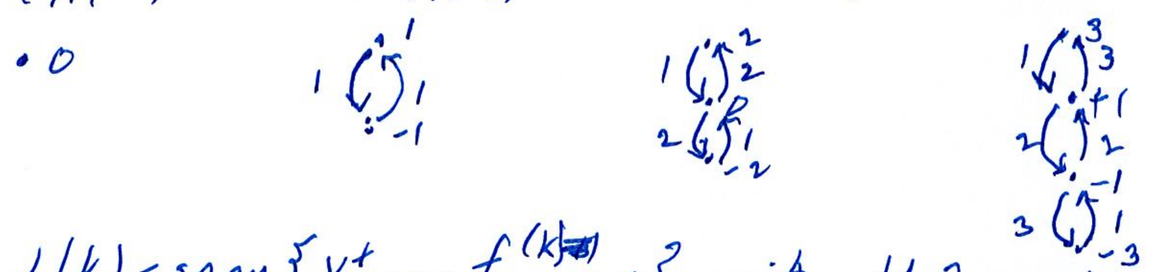
$$M(\mu)\supseteq N \quad\text{and}\quad M(\mu)/N=\mathrm{span}\{v^+,\dots,f^{(k+1)}v^+\}.$$

$k=0$ $\qquad$ $k=1$ $\qquad$ $k=2$



$M(0)\supseteq M(-2)$ $\qquad$ $M(1)\supseteq M(-3)$ $\qquad$ $M(2)\supseteq M(-4)$ $\qquad$ $M(3)\supseteq M(-5)$

$M(0)/M(-2)$ $\qquad$ $M(1)/M(-3)$ $\qquad$ $M(2)/M(-4)$ $\qquad$ $M(3)/M(-5)$



and $L(k)=\mathrm{span}\{v^+,\dots,f^{(k)}v^+\}$ with $U\mathfrak{sl}_2$-action given by

$$f=\begin{pmatrix}0&&&&&\\1&0&&&&\\&2&0&&&\\&&3&\ddots&&\\&&&&0&\\&&&&k&0\end{pmatrix}\quad h=\begin{pmatrix}k&&&&\\&k-2&&&\\&&\ddots&&\\&&&-(k-2)&\\&&&&-k\end{pmatrix}\quad e=\begin{pmatrix}0&k&&&&\\&0&k-1&&&\\&&\ddots&\ddots&&\\&&&0&2&\\&&&&0&1\\&&&&&0\end{pmatrix}$$