Representation Theory, Lecture 25, 22 September 2015

Univ. of Melbourne

$\mathfrak{g}$ = complex reductive Lie algebra

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$\mathfrak{b}$ = Borel subalgebra (maximal solvable)

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$\mathfrak{h}$ = Cartan subalgebra (maximal abelian)

Example $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}) = M_n(\mathbb{C})$

$\mathfrak{b} = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$ and $\mathfrak{h} = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$

$$\mathfrak{g} = \left(\bigoplus_{\varepsilon_i - \varepsilon_j \in R^+} \mathfrak{g}_{-(\varepsilon_i - \varepsilon_j)} \right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\varepsilon_i - \varepsilon_j \in R^+} \mathfrak{g}_{\varepsilon_i - \varepsilon_j} \right)$$

where

$$R^+ = \{ \varepsilon_i - \varepsilon_j \mid i,j \in \{1,2,\ldots,n\},\ i<j \}$$ and

$\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{C} E_{ij}$, with E_{ij} has 1 in the (i,j) entry and 0 elsewhere.

So

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$

and the simple roots are

$$\alpha_1 = \varepsilon_1 - \varepsilon_2,\ \alpha_2 = \varepsilon_2 - \varepsilon_3,\ \ldots,\ \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n$$

since

$$[\mathfrak{g}_{\varepsilon_i - \varepsilon_j}, \mathfrak{g}_{\varepsilon_j - \varepsilon_k}] = \mathfrak{g}_{\varepsilon_i - \varepsilon_k} \quad \text{for } i<j<k.$$

Example $\mathfrak{g} = \mathfrak{sl}_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a+d=0 \right\}$

$\mathfrak{b} = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a+d=0 \right\}$ and $\mathfrak{h} = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a+d=0 \right\}$.

Then

$$\mathfrak{g} = \mathfrak{g}_{-\alpha} \oplus \mathfrak{h} \oplus \mathfrak{g}_\alpha \text{ with}$$

$$\mathfrak{g}_{-\alpha} = \mathbb{C}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathfrak{h} = \mathbb{C}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathfrak{g}_\alpha = \mathbb{C}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Then the simple root is $\alpha_1 = \varepsilon_1 - \varepsilon_2$. and

$\mathfrak{g} = \mathfrak{sl}_2$ is generated by

$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ with relations

$[e,f] = h$, and $[h,e] = 2e$ and $[h,f] = -2f$

so that $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The enveloping algebra $U\mathfrak{sl}_2$ is the algebra given by generators e, f, h with relations

$$ef = fe + h, \quad eh = he - 2e, \quad fh = hf + 2f$$

A $\mathfrak{g}$-module is a $U\mathfrak{sl}_2$-module.

Let M be a finite dimensional $\mathfrak{g}$-module.

$$M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu \quad \text{where } M_\mu = \{m \in M \mid \text{if } h \in \mathfrak{h} \text{ then } hm = \mu(h)m\}$$

Let $R = \{\text{roots}\}$.

Let $\alpha \in R$ and let $e_\alpha \in \mathfrak{g}_\alpha$. Then, if $h \in \mathfrak{h}$ then

$$\begin{aligned} h e_\alpha m &= ([h, e_\alpha] + e_\alpha h) m \\ &= \alpha(h) e_\alpha m + e_\alpha h m \\ &= \alpha(h) e_\alpha m + e_\alpha \mu(h) m \\ &= (\alpha(h) + \mu(h)) e_\alpha m \\ &= (\alpha + \mu)(h) e_\alpha m. \end{aligned}$$

So $e_\alpha m \in M_{\alpha+\mu}$.

Define an order $\mathfrak{h}^*$ by

$\alpha + \mu > \mu$ for $\alpha \in R^+$. (dominance order).

A <u>highest weight</u> of M is $\mu \in \mathfrak{h}^*$ such that if $e_\alpha \in \mathfrak{n}^+$ then $e_\alpha M_\mu = 0$.

A <u>highest weight vector</u> is a vector $m \in M_\mu$

Let $\mu \in \mathfrak{h}^*$. The <u>Verma module</u> of highest weight μ is

$$M(\mu) = U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}_\mu, \quad \text{where}$$

$\mathbb{C}_\mu = \text{span}\{v^+\}$ with $ev^+ = 0$ and $hv^+ = \mu(h)v^+$ for $e \in \mathfrak{n}^+$ and $h \in \mathfrak{h}$.

Thus
$$M(\mu) = \mathcal{U}v^+, \quad \text{where } \mathcal{U} = \mathcal{U}\mathfrak{g}.$$

Since
$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^- \quad \text{then} \quad \mathcal{U} = \mathcal{U}^- \mathcal{U}_0 \mathcal{U}^+$$

and
$$M(\mu) = \mathcal{U}v^+ = \mathcal{U}^- \mathcal{U}_0 \mathcal{U}^+ v^+ = \mathcal{U}^- v^+.$$

<u>Note:</u>
$\mathcal{U} = \mathcal{U}^- \mathcal{U}_0 \mathcal{U}^+$ (where $\mathcal{U}^- = \mathcal{U}\mathfrak{n}^-$, $\mathcal{U}_0 = \mathcal{U}\mathfrak{h}$, $\mathcal{U}^+ = \mathcal{U}\mathfrak{n}^+$) is a version of the

<u>Poincaré-Birkhoff-Witt Theorem</u>:

If $\mathfrak{g}$ has basis $\{b_1, b_2, \ldots\}$ then

$\mathcal{U}\mathfrak{g}$ has basis $\left\{ b_1^{m_1} b_2^{m_2} \cdots b_\ell^{m_\ell} \;\middle|\; \begin{array}{l} \ell \in \mathbb{Z}_{>0} \text{ and} \\ m_1, m_2, \ldots, m_\ell \in \mathbb{Z}_{\geq 0} \end{array} \right\}$

So, in our case

if $\mathfrak{n}^-$ has basis $\{f_{\beta_1}, \ldots, f_{\beta_N}\}$

$\mathfrak{h}$ has basis $\{h_{\alpha_1^\vee}, \ldots, h_{\alpha_n^\vee}\}$

$\mathfrak{n}^+$ has basis $\{e_{\beta_1}, \ldots, e_{\beta_N}\}$

then $\mathcal{U}^-$ has basis $\{f_{\beta_1}^{m_1} \cdots f_{\beta_N}^{m_N} \mid m_1, \ldots, m_N \in \mathbb{Z}_{\geq 0}\}$

$\mathfrak{h}$ has basis $\{h_{\alpha_1^\vee}^{k_1} \cdots h_{\alpha_n^\vee}^{k_n} \mid k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0}\}$

$\mathcal{U}^+$ has basis $\{e_{\beta_1}^{n_1} \cdots e_{\beta_N}^{n_N} \mid n_1, \ldots, n_N \in \mathbb{Z}_{\geq 0}\}$.