Representation Theory lecture 24, 18 September 2015 (1)
Univ. of Melbourne

Let G be a connected complex reductive algebraic group.

$\mathfrak{g} = \mathrm{Lie}(G)$ the Lie algebra of G

$\mathfrak{h} = \mathrm{Lie}(T)$, where T is a maximal torus

G

$\cup|$

B a Borel subgroup

$\cup|$

T a maximal torus

$\mathfrak{g}$

$\cup|$

$\mathfrak{b}$ a Borel subalgebra

$\cup|$

$\mathfrak{h}$ a Cartan subalgebra

A <u>root</u> of $\mathfrak{g}$ is a nonzero weight of the adjoint representation of $\mathfrak{g}$.

As $\mathfrak{h}$-modules, $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \right)$

where $R = \{\text{roots of } \mathfrak{g}\}$, and

$\mathfrak{g}_\alpha = \{ x \in \mathfrak{g} \mid \text{if } h \in \mathfrak{h} \text{ then } [h,x] = \alpha(h)x \}$

A <u>positive root</u> is a nonzero weight of the adjoint representation of $\mathfrak{b}$.

As $\mathfrak{h}$-modules, $\mathfrak{b} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha \right)$

where $R = \{\text{positive roots of } \mathfrak{g}\}$

Example $G = GL_n(\mathbb{C})$, $\mathfrak{g} = M_n(\mathbb{C})$

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{i \neq j} \mathfrak{g}_{\varepsilon_i - \varepsilon_j} \right)$$

where $\mathfrak{h} = \{\text{diagonal matrices}\}$

$\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{C} E_{ij}$, where E_{ij} has 1 in (i,j) entry and 0 elsewhere

and $\varepsilon_i : \mathfrak{h} \longrightarrow \mathbb{C}$

$\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \longmapsto a_i$. Note that if $h = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \in \mathfrak{h}$

then $[h, E_{ij}]$

$$= \left[\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}, E_{ij} \right] = a_i E_{ij} - E_{ij} a_j = (a_i - a_j) E_{ij}$$

$$= (\varepsilon_i - \varepsilon_j) \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \cdot E_{ij} = (\varepsilon_i - \varepsilon_j)(h) E_{ij}.$$

So $R = \{ \varepsilon_i - \varepsilon_j \mid i,j \in \{1, 2, \ldots, n\} \text{ with } i \neq j \}$

and $R^+ = \{ \varepsilon_i - \varepsilon_j \mid i,j \in \{1, 2, \ldots, n\} \text{ with } i < j \}$.

$[\mathfrak{b}, \mathfrak{b}] = \mathfrak{n}^+$ and

$$\mathfrak{b} \supseteq [\mathfrak{b}, \mathfrak{b}] \supseteq [[\mathfrak{b},\mathfrak{b}],[\mathfrak{b},\mathfrak{b}]] \supseteq \cdots$$

is the <u>derived series</u> of $\mathfrak{b}$. By definition $\mathfrak{b}$ is <u>solvable</u>, which means that this series is finite.

Then

$$\mathfrak{g} = \Big(\bigoplus_{\alpha \in R^+} \mathfrak{g}_{-\alpha}\Big) \oplus \mathfrak{h} \oplus \Big(\bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha\Big)$$

with

$$\mathfrak{n}^- = \Big(\bigoplus_{\alpha \in R^+} \mathfrak{g}_{-\alpha}\Big) \quad \text{and} \quad \mathfrak{n}^+ = \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha$$

Note that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\beta+\gamma}$

The <u>simple roots</u> $\alpha_1, \ldots, \alpha_n$ are the roots $\alpha \in R^+$ such that there does not exist $\beta, \gamma \in R$ with $\beta + \gamma = \alpha$.

The <u>point is</u> that

$\mathfrak{n}^+$ is generated by $e_1, \ldots, e_n$

$\mathfrak{n}^-$ is generated by $f_1, \ldots, f_n$

$\mathfrak{g}$ is generated by $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$.

<u>HW</u> What are the relations??

<u>HW</u> Let $e_\alpha \in \mathfrak{g}_\alpha$. Then there exists $f_\alpha \in \mathfrak{g}_{-\alpha}$ and $h_{\alpha^\vee} \in \mathfrak{h}$ with

$$[e_\alpha, f_\alpha] = h_{\alpha^\vee}, \quad [h_{\alpha^\vee}, e_\alpha] = 2e_\alpha, \quad [h_{\alpha^\vee}, f_\alpha] = -2f_\alpha.$$

<u>HW</u> There is a Lie algebra homomorphism

$$\mathfrak{sl}_2 \to \mathfrak{g}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \longmapsto e_\alpha$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \longmapsto f_\alpha$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \longmapsto h.$$