Representation Theory, Lecture 20, 10 Sept. 2015

Univ. of Melbourne

Borel subgroups

Theorem Let G be a connected linear algebraic group.

All Borel subgroups are conjugate in G.

Let G be a connected linear algebraic group.

The set of connected closed solvable subgroups of G is ordered by inclusion.

A Borel subgroup is a maximal connected closed solvable subgroup of G.

Example: If $G = GL_n(\mathbb{C})$ then

$$B = \left\{ \begin{pmatrix} a_{11} & \ddots & a_{ij} \\ 0 & & a_{nn} \end{pmatrix} \in GL_n(\mathbb{C}) \right\}$$ is a Borel subgroup of G

Maximal compact subgroups

Theorem Let G be a connected Lie group.

All maximal compact subgroups of G are conjugate in G.

Let G be a connected Lie group.

The set of compact subgroups of G is ordered by inclusion

Example If $G = GL_n(\mathbb{C})$ then

$$U_n(\mathbb{C}) = \{ g \in GL_n(\mathbb{C}) \mid g\bar{g}^t = 1 \}$$

is a maximal compact subgroup.

If $G = SL_2(\mathbb{R})$ then $SO_2(\mathbb{R}) = \{ g \in SL_2(\mathbb{R}) \mid g\bar{g}^t = 1 \}$

is a maximal compact subgroup.

Maximal tori and Cartan subalgebras

Theorem [Bou, Lie Ch. IX §2 No 1 Theorem 1 and No 2 Theorem 2]

~~(a) Let $\mathfrak{g}$ be a compact Lie algebra~~

Let K be a connected compact Lie group.

~~Any~~ Maximal tori are conjugate in G.

The Lie algebras of the maximal tori are the Cartan subalgebras of $\mathfrak{k} = \mathrm{Lie}(K)$.

A torus in K is a subgroup H isomorphic to

$$\underbrace{S^1 \times S^1 \times \cdots \times S^1}_{r \text{ times}} \text{ with } r \in \mathbb{Z}_{>0}$$

The set of tori in K is ordered by inclusion

Example $U_1(\mathbb{C}) = \{ z \in \mathbb{C}^\times \mid z\bar{z} = 1 \} = \{ e^{i\theta} \mid 0 \le \theta < 2\pi \} = S^1$.

If $K = U_n(\mathbb{C})$ then

$$H = \left\{ \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \middle| a_i \in U_1(\mathbb{C}) \right\}$$ is a maximal torus in K.

Sylow subgroups

Theorem Let G be a finite group.

All p-Sylow subgroups are conjugate in G.

Let $p \in \mathbb{Z}_{>0}$ be prime.

A p-group is a finite group with order a power of p.

Let G be a finite group.

The set of p-subgroups of G is ordered by inclusion.

A p-Sylow subgroup is a maximal p-subgroup of G

Example Let $\mathbb{F}_q$ be the field with q elements.

Then

$$\begin{aligned} \mathrm{Card}(GL_n(\mathbb{F}_q)) &= (q^n-1)(q^n-q)\cdots(q^n-q^{n-1}) \\ &= q^{(n-1)+(n-2)\cdots+1}(q^n-1)(q^{n-1}-1)\cdots(q-1) \\ &= q^{\binom{n}{2}}(q-1)^n\left(\frac{q^n-1}{q-1}\right)\left(\frac{q^{n-1}-1}{q-1}\right)\cdots\left(\frac{q-1}{q-1}\right) \\ &= \left(\sum_{w\in S_n} q^{\ell(w)}\right)(q-1)^n\, q^{1+2+\cdots+n-1} \end{aligned}$$

So, if $q=p$ then $G=GL_n(\mathbb{F}_q)$

$$U^+ = \left\{ \begin{pmatrix} 1 & & a_{ij} \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \,\middle|\, a_{ij}\in\mathbb{F}_p \right\}$$ is a p-Sylow subgroup of G.

G acts on G/B.

Then B is the stabilizer of B in G/B.

G is certainly transitive on G/B.

Certainly $\quad gBg^{-1} \subseteq \mathrm{Stab}(gB)$

So $\quad \left\{ \begin{matrix} \text{conjugates} \\ \text{of } B \end{matrix} \right\} \simeq G/B$. since G acts on $\left\{ \begin{matrix} \text{Borel} \\ \text{subgroups} \end{matrix} \right\}$ by conjugation

In each case we get a nice homogeneous space.

G acts on G/B the flag variety

K acts on K/H the flag variety

G acts on G/K the upper half plane

G acts on $\left\{ \begin{matrix} p\text{-Sylow} \\ \text{subgroups} \end{matrix} \right\} \simeq$ ~~G/U~~ G/U and $\mathrm{Card}(G/U) = 1 \bmod p$.