Representation theory: Lecture 1 28 July 2015
Univ. of Melbourne

Housekeeping: Web page
Consultation hours / Michèle Vergne Cuming Theatre
Handbook entry - time commitment Billey + Ram
Assignment 1.

Proof Machine Lecture / Grammar page / examples

What is representation theory?

An algebra is a vector space A with a function

$$\begin{aligned} A\otimes A &\to A \\ (a,b) &\mapsto ab \end{aligned}$$ such that

(1) If $c_1, c_2 \in \mathbb{F}$ and $a_1, a_2, a_3 \in A$ then
$$(c_1a_1 + c_2a_2)a_3 = c_1(a_1a_3) + c_2(a_2a_3)$$
(2) If $c_1, c_2 \in \mathbb{F}$ and $a_1, a_2, a_3 \in A$ then
$$a_1(c_1a_2 + c_2a_3) = c_1(a_1a_2) + c_1(a_1a_3)$$
(3) If $a_1, a_2, a_3 \in \mathbb{F}$ then $(a_1a_2)a_3 = a_1(a_2a_3)$
(4) There exists $1 \in A$ such that
if $a \in A$ then $1 \cdot a = a$ and $a \cdot 1 = a$

An <u>A-module</u> is a vector space M with a function

$$A \otimes M \to M \qquad \text{such that}$$
$$(a, m) \mapsto am$$

(1) If $c_1, c_2 \in \mathbb{F}$, $a_1, a_2 \in A$ and $m \in M$ then

$$(c_1 a_1 + c_2 a_2) m = c_1 (a_1 m) + c_2 (a_2 m)$$

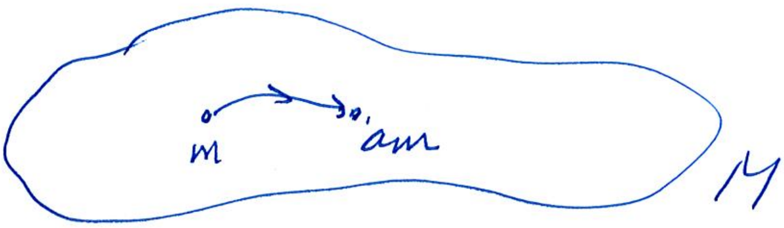
(2) If $c_1, c_2 \in \mathbb{F}$, $a \in A$ and $m_1, m_2 \in M$ then

$$a(c_1 m_1 + c_2 m_2) = c_1 (a m_1) + c_2 (a m_2)$$

(3) If $a_1, a_2 \in A$ and $m \in M$ then

$$a_1 (a_2 m) = (a_1 a_2) m$$

(4) If $m \in M$ then $1 \cdot m = m$.



Let M be an A-module.

The <u>representation</u> associated to M is the algebra homomorphism

$$\rho : A \to \mathrm{End}(M)$$
$$a \mapsto a_M$$

where a_M is the linear transformation $a_M : M \to M$ given by

$$a_M(m) = am$$

If we choose a basis $\{b_1, \dots, b_d\}$ of M then

$$\rho : A \longrightarrow M_d(\mathbb{C})$$
$$a \longmapsto (a_{ij})$$

where $a b_i = \sum_{j=1}^{d} a_{ji} b_j$

Let M and N be A-modules.

A <u>morphism</u> from M to N is a linear transformation

$f : M \to N$ such that

if $a \in A$ and $m \in M$ then $f(am) = af(m)$.

<u>Categories so far:</u>

Vector spaces and linear transformations

Rings and ring homomorphisms

Algebras and algebra homomorphisms

Fields and field homomorphisms

A-modules and A-module homomorphisms

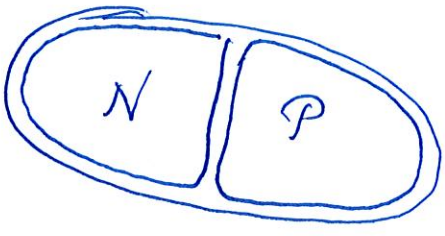
<u>Representation theory</u> is the study of the category of A-modules.

A <u>simple A-module</u> is an A-module M such that

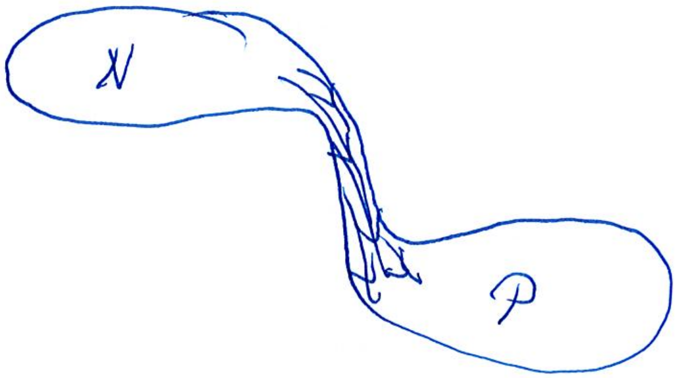
if N is a submodule of M and $N \neq 0$ then $N = M$.

An <u>indecomposable A-module</u> is an A-module M such that

there do not exist submodules N and P with $N \neq 0$, $P \neq 0$ and $M = N \oplus P$.



$M = N \oplus P$



$$0 \to P \to M \to N \to 0$$

$P = \ker \varphi$ and $N \cong M/P$

but $M \ncong N \oplus P$ as A-modules since N is not a submodule