Representation Theory Lecture 19, 8 September 2015
Univ. of Melbourne

$G = GL_n(\mathbb{C})$ with $T = \left\{ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \;\middle|\; a_1, a_2, \ldots, a_n \in \mathbb{C}^\times \right\}$

Then

$$\mathfrak{h}_{\mathbb{Z}} = \mathrm{Hom}(\mathbb{C}^\times, T) \simeq \mathbb{Z}\text{-span}\{\varepsilon_1^\vee, \varepsilon_2^\vee, \ldots, \varepsilon_n^\vee\}$$

$$\begin{array}{ccc} h_{\varepsilon_i^\vee} : \mathbb{C}^\times \to T & \longleftarrow & \varepsilon_i^\vee \\ z \mapsto \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & z & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} & & \end{array}$$

so that, if $\mu^\vee = \mu_1 \varepsilon_1^\vee + \cdots + \mu_n \varepsilon_n^\vee$ then

$$h_{\mu^\vee}(z) = h_{\varepsilon_1^\vee}(z)^{\mu_1} \cdots h_{\varepsilon_n^\vee}(z)^{\mu_n}.$$

Then

$$\mathfrak{h}_{\mathbb{Z}}^* = \mathrm{Hom}(T, \mathbb{C}^\times) \simeq \mathbb{Z}\text{-span}\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$$

$$\begin{array}{ccc} X^{\varepsilon_i} : T \to \mathbb{C}^\times & \longleftarrow & \varepsilon_i \\ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \mapsto a_i & & \end{array}$$

so that, if $\lambda = \lambda_1 \varepsilon_1 + \cdots \lambda_n \varepsilon_n$ then

$$X^\lambda \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} = a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n}.$$

Composition gives a pairing

$$\begin{array}{rcl} \mathfrak{h}_{\mathbb{Z}} \times \mathfrak{h}_{\mathbb{Z}}^* = \mathrm{Hom}(\mathbb{C}^\times, T) \times \mathrm{Hom}(T, \mathbb{C}^\times) & \longrightarrow & \mathrm{Hom}(\mathbb{C}^\times, \mathbb{C}^\times) \simeq \mathbb{Z} \\ (h, X) & \longmapsto & X \circ h : \mathbb{C}^\times \to \mathbb{C}^\times \\ (h_{\mu^\vee}, X^\lambda) & \longmapsto & z^{\langle \mu^\vee, \lambda \rangle} \end{array}$$

where $\langle \varepsilon_i^\vee, \varepsilon_j \rangle = \delta_{ij}$.

The <u>Weyl group</u> is $W_0 = N_G(T)/Z_G(T)$.

$$Z_G(T) = T = \left\{ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \,\middle|\, a_1, \ldots, a_n \in \mathbb{C}^\times \right\}$$

$N_G(T) = \left\{ \begin{array}{l} n\times n \text{ matrices with exactly one nonzero} \\ \text{entry in each row and each column} \\ \text{and nonzero entries in } \mathbb{C}^\times \end{array} \right\}$

$$= \left\{ w \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \,\middle|\, w \in S_n,\ a_1, \ldots, a_n \in \mathbb{C}^\times \right\}.$$

So

$$W_0 = N_G(T)/T = \{ wT \mid w \in S_n \} \cong S_n.$$

Then $W_0 = S_n$ acts $\mathbb{Z}$-linearly on

$\mathfrak{h}_{\mathbb{Z}} = \mathbb{Z}\text{span}\{\varepsilon_1^\vee, \ldots, \varepsilon_n^\vee\}$ by permuting $\varepsilon_1^\vee, \ldots, \varepsilon_n^\vee$

$\mathfrak{h}_{\mathbb{Z}}^* = \mathbb{Z}\text{-span}\{\varepsilon_1, \ldots, \varepsilon_n\}$ by permuting $\varepsilon_1, \ldots, \varepsilon_n$.

So

$$w\varepsilon_i^\vee = \varepsilon_{w(i)}^\vee \quad \text{and} \quad w\varepsilon_j = \varepsilon_{w(j)}$$

for $i, j \in \{1, 2, \ldots n\}$ and $w \in S_n$.

Complex one parameter subgroups of G are group homomorphisms $\mathbb{C} \to G$.

Let E_{ij} be the $n\times n$ matrix with 1 in the (i,j) entry and zero elsewhere. Define

$$\begin{aligned} x_{ij}: \mathbb{C} &\to GL_n(\mathbb{C}) \\ t &\mapsto x_{ij}(t) \end{aligned} \quad \text{where } x_{ij}(t) = 1 + tE_{ij} = \begin{pmatrix} 1 & & t \\ & \ddots & \\ & & 1 \end{pmatrix}$$

Then $x_{ij}(t)x_{ij}(s) = x_{ij}(t+s)$ so that x_{ij} is a group homomorphism.

Define

$$\begin{aligned} \mathbb{C} \to \mathbb{C}^\times &\to GL_n(\mathbb{C}) \\ t \mapsto e^t &\mapsto h_{\varepsilon_i^\vee}(e^t) \end{aligned} \quad \text{so that } h_i(e^t) = 1 + (e^t - 1)E_{ii}.$$

Then $h_{\varepsilon_i^\vee}(e^t)h_{\varepsilon_i^\vee}(e^s) = h_{\varepsilon_i}(e^{t+s})$.

Note that

$$\left(\frac{d}{dt} x_{ij}(t)\right)\Big|_{t=0} = E_{ij} \quad \text{and} \quad \left(\frac{d}{dt}\left(h_{\varepsilon_i^\vee}(e^t)\right)\right)\Big|_{t=0} = e^0 E_{ii} = E_{ii}.$$

The Lie algebra of $GL_n(\mathbb{C})$ (as a complex Lie group) is

$$\mathfrak{gl}_n(\mathbb{C}) = \mathbb{C}\text{-span}\{E_{ij} \mid i,j \in \{1,\dots,n\}\} = M_n(\mathbb{C}).$$

The exponential map is

$$\begin{aligned} \mathfrak{gl}_n &\to GL_n(\mathbb{C}) \\ X &\mapsto e^{tX} \end{aligned} \quad \text{where } e^A = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

The Lie algebra $\mathfrak{g} = \mathfrak{gl}_n$ has <u>Cartan subalgebra</u>

$$\mathfrak{h}_{\mathbb{C}} = \mathbb{C}\text{span}\{E_{11}, \ldots, E_{nn}\} = \mathbb{C} \otimes_{\mathbb{Z}} \mathfrak{h}_{\mathbb{Z}}$$

and the 1-dimensional representations of $\mathfrak{h}_{\mathbb{C}}$ are

$$\mathfrak{h}_{\mathbb{C}}^* = \text{Hom}(\mathfrak{h}_{\mathbb{C}}, \mathbb{C}) = \mathbb{C}\text{-span}\{\varepsilon_1, \ldots, \varepsilon_n\} = \mathbb{C} \otimes_{\mathbb{Z}} \mathfrak{h}_{\mathbb{Z}}^*$$

where

$$\begin{array}{rcl} \varepsilon_i : \mathfrak{h}_{\mathbb{C}} & \longrightarrow & \mathbb{C} \\ \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} & \longmapsto & a_i \end{array} \quad \text{for } i \in \{1, 2, \ldots, n\}.$$

If M is a $\mathfrak{g}$-module then

$$M = \bigoplus_{\mu \in \mathfrak{h}_{\mathbb{C}}^*} M_\mu \quad \text{where } M_\mu = \left\{ m \in M \;\middle|\; \begin{array}{l} \text{if } x \in \mathfrak{h}_{\mathbb{C}} \text{ then} \\ xm = \mu(x) m \end{array} \right\}$$

The $\mu \in \mathfrak{h}_{\mathbb{C}}^*$ such that $M_\mu \neq 0$ are the <u>weights</u> of M.