Representation Theory Lecture 11, 20 August 2015

Univ. of Melbourne

Let λ be a partition with k-boxes.

A <u>standard tableau of shape λ</u> is a filling of the boxes of λ with $1, 2, \ldots, k$ such that

(a) the rows are increasing left to right

(b) the columns are increasing top to bottom.

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}$$

are the standard tableaux of shape $2\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3$.

Let

$$S(\lambda) = \{\text{standard tableaux of shape } \lambda\}.$$

If $T \in S(\lambda)$ define

$$s_i T = (T \text{ except } i \text{ and } i+1 \text{ are switched in } T)$$

and set

$$\tilde{s}_i T = \begin{cases} s_i T, & \text{if } s_i T \text{ is a standard tableau} \\ 0, & \text{otherwise} \end{cases}$$

The action of $\tilde{s}_1, \ldots, \tilde{s}_{k-1}$ on $S(\lambda)$ makes $S(\lambda)$ into an S_k-crystal.

$$B^{\otimes 2} = B(\square\square) \oplus B\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right) \quad \text{where}$$

$$B(\square\square) = \{p_{i_1}\otimes p_{i_2} \mid i_1 \geq i_2\} \quad \text{and} \quad p_{i_1}\otimes p_{i_2} \longleftrightarrow \boxed{i_2}\boxed{i_1}$$

$$B\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right) = \{p_{i_1}\otimes p_{i_2} \mid i_1 < i_2\}. \quad \text{and} \quad p_{i_1}\otimes p_{i_2} \longleftrightarrow \begin{smallmatrix}\boxed{i_1}\\\boxed{i_2}\end{smallmatrix}$$

$$B^{\otimes 3} = B(\square\square\square) \oplus B\left(\begin{smallmatrix}\square\square\\\square\end{smallmatrix}\right) \oplus B\left(\begin{smallmatrix}\square\square\\\square\end{smallmatrix}\right) \oplus B\left(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}\right)$$

$$B(\square\square\square) = \{p_{i_1}\otimes p_{i_2}\otimes p_{i_3} \mid i_1 \geq i_2 \geq i_3\}$$

$$B\left(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}\right) = \{p_{i_1}\otimes p_{i_2}\otimes p_{i_3} \mid i_1 < i_2 < i_3\}$$

$$B\left(\begin{smallmatrix}\square\square\\\square\end{smallmatrix}\right) \otimes \begin{smallmatrix}\boxed{1}\boxed{2}\\\boxed{3}\end{smallmatrix} = \left\{\begin{matrix} p_1p_1p_2,\ p_2p_1p_2,\ p_2p_1p_3,\ p_3p_1p_3,\ p_3p_2p_3 \\ p_1p_1p_3,\ p_3p_1p_2,\ p_2p_2p_3, \end{matrix}\right\}$$

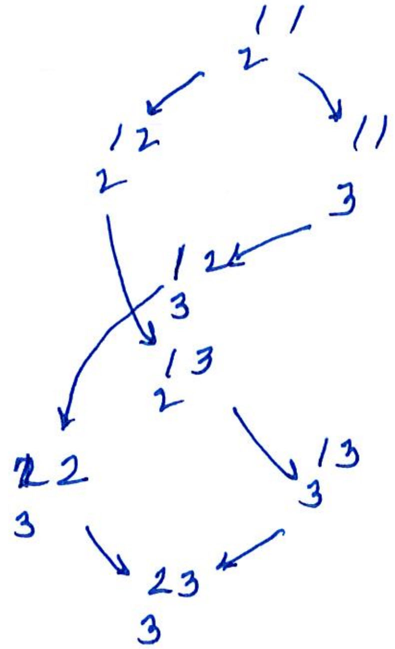
corresponding to $\begin{smallmatrix}1&1\\2\end{smallmatrix}$, $\begin{smallmatrix}1&2\\2\end{smallmatrix}$, $\begin{smallmatrix}1&2\\3\end{smallmatrix}$, $\begin{smallmatrix}1&3\\3\end{smallmatrix}$, $\begin{smallmatrix}2&3\\3\end{smallmatrix}$.
$\begin{smallmatrix}1&1\\3\end{smallmatrix}$, $\begin{smallmatrix}1&3\\2\end{smallmatrix}$, $\begin{smallmatrix}2&2\\3\end{smallmatrix}$,

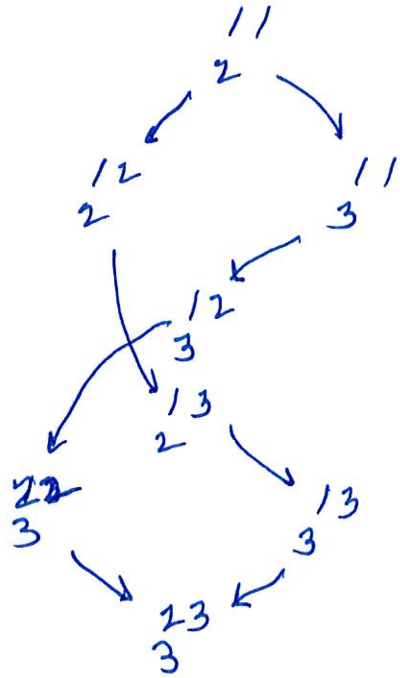
So $$B\left(\begin{smallmatrix}\square\square\\\square\end{smallmatrix}\right) \otimes \begin{smallmatrix}1&2\\3\end{smallmatrix} = \left\{p_{i_1}\otimes p_{i_2}\otimes p_{i_3} \,\middle|\, \begin{matrix} i_1 \geq i_2 \\ i_2 < i_3 \end{matrix}\right\}$$

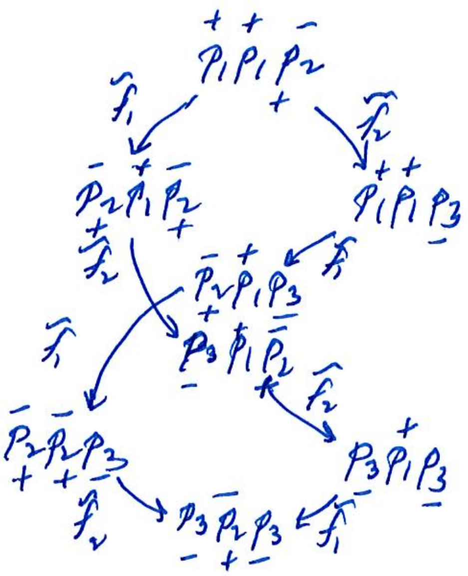
$$B\left(\begin{smallmatrix}\square\square\\\square\end{smallmatrix}\right) \otimes \begin{smallmatrix}\boxed{1}\boxed{3}\\\boxed{2}\end{smallmatrix} = \left\{\begin{matrix} p_1p_2p_1,\ p_1p_2p_2,\ p_2p_3p_1,\ p_1p_3p_3,\ p_2p_3p_3 \\ p_1p_3p_1,\ p_1p_3p_2,\ p_2p_3p_2, \end{matrix}\right\}$$

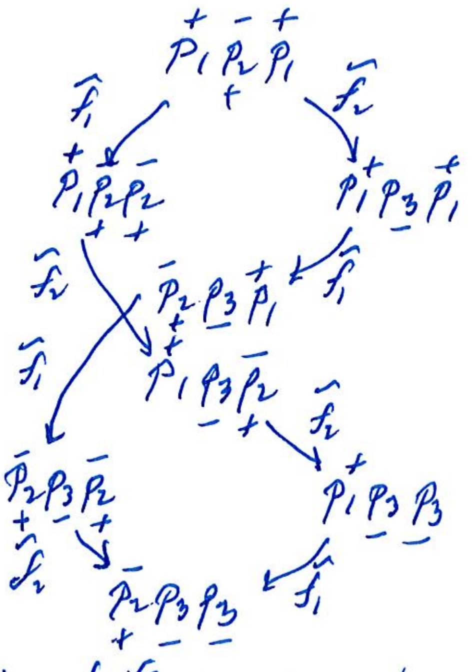
So $$B\left(\begin{smallmatrix}\square\square\\\square\end{smallmatrix}\right) \otimes \begin{smallmatrix}1&3\\2\end{smallmatrix} = \left\{p_{i_1}\otimes p_{i_2}\otimes p_{i_3} \,\middle|\, \cancel{\begin{matrix} i_1 < i_2 \\ i_2 > i_3 \end{matrix}} \ \begin{matrix} i_2 > i_1 \\ i_3 \leq i_1 \text{ or } i_1 < i_3 \leq i_2 \end{matrix}\right\}$$

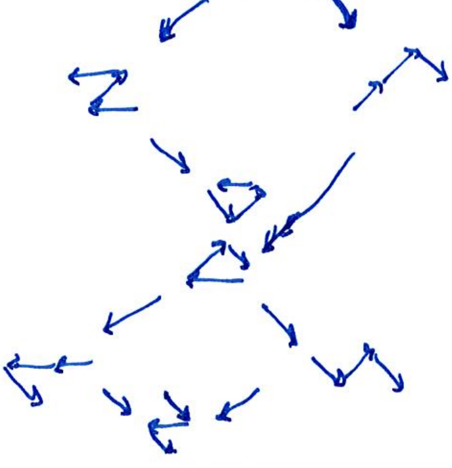
The bicrystal $B(\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}) \otimes S(\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix})$ is











Action of $\tilde{f}_i$ is given by

After pairing change the leftmost + to −.

Action of $\tilde{s}_2$ is given by

$$\tilde{s}_2(p_{i_1}\tilde{p}_{i_2}\tilde{p}_{i_3}) = \begin{cases} p_{i_1}p_{i_3}p_{i_2}, & \text{if } i_3 \le i_1 < i_2 \text{ or } i_2 \le i_1 < i_3 \\ p_{i_2}p_{i_1}p_{i_3}, & \text{if } i_1 \le i_3 \le i_2 \text{ or } i_2 \le i_3 \le i_1 \end{cases}$$

Action of s_2 is given by

$$s_2(p_{i_1} p_{i_2} p_{i_3}) = \begin{cases} p_{i_1} p_{i_3} p_{i_2}, & \text{if } i_2 \le i_1 < i_3 \text{ or } i_3 \le i_1 < i_2 \\ p_{i_2} p_{i_1} p_{i_3}, & \text{if } i_1 < i_3 \le i_2 \text{ or } i_2 < i_3 \le i_1 \end{cases}$$

$$= \begin{cases} s_2(p_{i_1} p_{i_2} p_{i_3}), & \text{if } i_2 \le i_1 < i_3 \text{ or } i_3 \le i_1 < i_2 \\ s_1(p_{i_1} p_{i_2} p_{i_3}), & \text{if } i_1 < i_3 \le i_2 \text{ or } i_2 < i_3 \le i_1 \\ 0, & \text{if } i_1 \ge i_2 \ge i_3 \\ 0, & \text{if } i_1 < i_2 < i_3 \end{cases}$$

$$= \begin{cases} s_2(p_{i_1} p_{i_2} p_{i_3}), & \text{if } i_2 \le i_1 < i_3 \text{ or } \cancel{i_1 < i_2 \le i_3}\ i_2 > i_1 \ge i_3 \\ s_1(p_{i_1} p_{i_2} p_{i_3}), & \text{if } i_1 < i_3 \le i_2 \text{ or } i_1 \ge i_3 > i_2. \\ 0, & \text{if } i_1 \ge i_2 \ge i_3 \\ 0 & \text{if } i_1 < i_2 < i_3 \end{cases}$$

$$= \begin{cases} s_2(p_{i_1} p_{i_2} p_{i_3}), & \text{if } i_2 > i_1 \ge i_3, \\ 0\ \cancel{s_1(p_{i_1} p_{i_2} p_{i_3})}, & \text{if } i_1 \ge i_2 \ge i_3, \\ s_1(p_{i_1} p_{i_2} p_{i_3}), & \text{if } i_1 \ge i_3 > i_2, \\ s_2(p_{i_1} p_{i_2} p_{i_3}), & \text{if } i_2 \le i_1 < i_3, \\ 0, & \text{if } i_1 < i_2 < i_3, \\ s_1(p_{i_1} p_{i_2} p_{i_3}), & \text{if } i_1 < i_3 \le i_2 \end{cases}$$