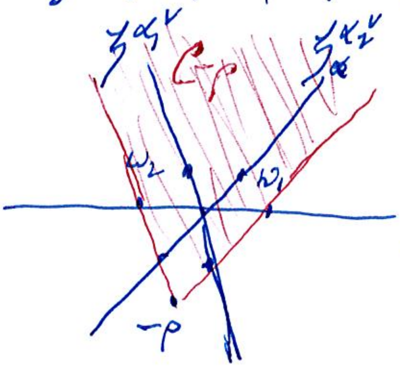
Representation Theory Lecture 10, 18 August 2015
Univ. of Melbourne (1)

SL_3 data review $\mathfrak{h}^*_{\mathbb{Z}} = \mathbb{Z}\text{span}\{\omega_1, \omega_2\}$

$W_0 = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1 \text{ and } s_1 s_2 s_1 = s_2 s_1 s_2 \rangle.$



$\mathfrak{h}^*_{\mathbb{R}} = \mathbb{R}\text{-span}\{\omega_1, \omega_2\}$

$\langle \lambda, \alpha_i^\vee \rangle$ = distance to $\mathfrak{h}^{\alpha_i^\vee}$ from λ.

GL_n-data $\mathfrak{h}^*_{\mathbb{Z}} = \mathbb{Z}\text{-span}\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$ with
W_0 acting by permuting $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ and $\mathbb{Z}$-linearly.
$\mathfrak{h}^*_{\mathbb{R}} = \mathbb{R}\text{-span}\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$ and

$C = \{\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \in \mathfrak{h}^*_{\mathbb{R}} \mid \lambda_i > \lambda_{i+1}\}$ and

$\rho = (n-1)\varepsilon_1 + (n-2)\varepsilon_2 + \cdots + 2\varepsilon_{n-2} + \varepsilon_{n-1}$ and

$\langle \lambda, \alpha_i^\vee \rangle = \lambda_i - \lambda_{i+1}$ if $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n$

provides the "distance from λ to $\mathfrak{h}^{\alpha_i^\vee}$".

GL_n-crystals are collections of paths $p : [0,1] \to \mathfrak{h}^*_{\mathbb{R}}$
with $p(0) = 0$ and $p(1) \in \mathfrak{h}^*_{\mathbb{Z}}$
which are closed under the root operators
$\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_{n-1}$ and $\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_{n-1}$

The defining representation of $GL_n(\mathbb{C})$

$GL_n(\mathbb{C})$ acts on $V = \mathbb{C}^n$ by matrix multiplication.

A maximal torus of $GL_n(\mathbb{C})$ is

$$T = \left\{ \begin{pmatrix} x_1 & & 0 \\ & x_2 & \\ 0 & & \ddots & \\ & & & x_n \end{pmatrix} \,\middle|\, x_i \in \mathbb{C}^\times \right\} \quad \text{and} \quad \begin{array}{rcl} X^{\varepsilon_i} : T & \longrightarrow & \mathbb{C}^\times \\ \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} & \longmapsto & x_i \end{array}$$

are homomorphism that generate $\mathfrak{h}^*_{\mathbb{Z}} = \mathrm{Hom}(T, \mathbb{C}^\times)$.

Let $[X^{\varepsilon_i}]$ denote the 1-dimensional T-module corresponding to X^{ε_i}. Then

$$V \cong [X^{\varepsilon_1}] \oplus \cdots \oplus [X^{\varepsilon_n}] \quad \text{as a } T\text{-module}.$$

(if $(e_1, e_2, \dots, e_n)$ is the standard basis of V then

$$t e_i = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} e_i = x_i e_i = X^{\varepsilon_i}\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} e_i = X^{\varepsilon_i}(t) e_i.$$)

The crystal corresponding to V is

$$B = \{p_1, p_2, \dots, p_n\}, \quad \text{where}$$

p_i is the straight line path to ε_i and the crystal graph of B is

$$p_1 \xrightarrow{\tilde{f}_1} p_2 \xrightarrow{\tilde{f}_2} \cdots \longrightarrow p_{n-1} \xrightarrow{\tilde{f}_{n-1}} p_n$$

Let $x_i = X^{\varepsilon_i}$. Then

$$\mathrm{char}(B) = X^{\varepsilon_1} + \cdots + X^{\varepsilon_n} = x_1 + \cdots + x_n = s_{\square}$$

Tensor products

Let G be a group and let M and N be G-modules. The tensor product $M\otimes N$ is the G-module given by

$$g(m\otimes n) = gm\otimes gn, \text{ for } m\in M, n\in N \text{ and } g\in G.$$

Let $\mathfrak{g}$ be a Lie algebra and let M and N be $\mathfrak{g}$-modules. The tensor product $M\otimes N$ is the $\mathfrak{g}$-module given by

$$x(m\otimes n) = xm\otimes n + m\otimes xn, \text{ for } m\in M, n\in N \text{ and } x\in\mathfrak{g}.$$

Let B_1 and B_2 be crystals. The tensor product is $B_1\otimes B_2 = B_1\times B_2 = \left\{ p\otimes q \;\middle|\; \begin{matrix} p\in B_1 \\ q\in B_2 \end{matrix} \right\}$ with action of the root operators given by

$$\tilde{e}_i(p\otimes q) = \begin{cases} \tilde{e}_i p\otimes q, & \text{if } d_i^+(p)\geq d_i^-(q) \\ p\otimes \tilde{e}_i q, & \text{if } d_i^+(p)< d_i^-(q) \end{cases}$$

$$\tilde{f}_i(p\otimes q) = \begin{cases} \tilde{f}_i p\otimes q, & \text{if } d_i^+(p)> d_i^-(q) \\ p\otimes \tilde{f}_i q, & \text{if } d_i^+(p)\leq d_i^-(q) \end{cases}$$

The crystal $B^{\otimes k} = \underbrace{B \otimes B \otimes \cdots \otimes B}_{k \text{ factors}}$

$$B^{\otimes k} = \{ p_{i_1} \otimes \cdots \otimes p_{i_k} \mid i_1, \ldots, i_k \in \{1, 2, \ldots, n\} \},$$

the set of <u>words of length k</u> from the alphabet $\{p_1, p_2, \ldots, p_n\}$.

Let

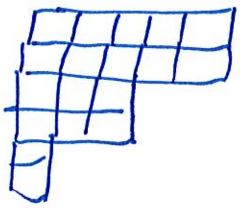
$$\lambda \in (C - \rho) \cap \mathfrak{h}^*_{\mathbb{Z}} = \{ \lambda = \lambda_1 \varepsilon_1 + \cdots + \lambda_n \varepsilon_n \mid \lambda_i \in \mathbb{Z},\ \lambda_i \geq \lambda_{i+1} \}.$$

$$\mu \in \mathfrak{h}^*_{\mathbb{Z}} = \{ \mu = \mu_1 \varepsilon_1 + \cdots + \mu_n \varepsilon_n \mid \mu_i \in \mathbb{Z} \}$$

Assume $\lambda_n \geq 0$. Identify λ with a <u>partition</u>

a collection of boxes in a corner where gravity pushes up and left

$$5\varepsilon_1 + 5\varepsilon_2 + 3\varepsilon_3 + 3\varepsilon_4 + \varepsilon_5 + \varepsilon_6 =$$



so that

$$\lambda_i = \# \text{ of boxes in row } i.$$

A <u>column strict tableau of shape λ and weight μ</u> is a filling of the boxes of λ with μ_1 1's, μ_2 2's, ..., μ_n n's such that

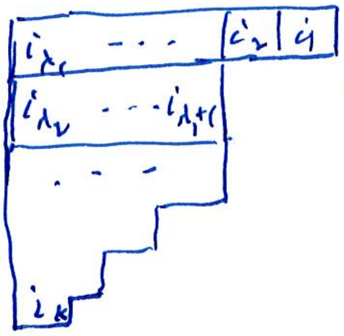
(a) the rows are weakly increasing left to right

(b) the columns are strictly increasing top to bottom

Let $B(\lambda) = \{\text{column strict tableaux of shape } \lambda\}$

Define

$$\mathrm{word}: B(\lambda) \hookrightarrow B^{\otimes k}$$



$$\longmapsto \quad p_{i_1} \otimes p_{i_2} \otimes \cdots \otimes p_{i_k}$$

i.e. $\mathrm{word}(p) = (\text{arabic reading of } p)$

Then $B(\lambda)$ is a subcrystal of $B^{\otimes k}$;

$$\mathrm{char}(B^{\otimes k}) = (x_1 + x_2 + \cdots + x_n)^k \quad \text{and}$$

$$s_\lambda = \mathrm{char}(B(\lambda)) = \sum_{p \in B(\lambda)} x_1^{\#1\text{'s in } p} x_2^{\#2\text{'s in } p} \cdots x_n^{\#n\text{'s in } p}$$

is the <u>Schur function</u> s_λ indexed by λ.

<u>Theorem</u>

$$s_\lambda = \frac{\det(x_i^{\lambda_j + n - j})}{\det(x_i^{n-j})}$$

$$= \frac{\sum_{w \in S_n} \det(w)\, w(x_1^{\lambda_1 + n-1} x_2^{\lambda_2 + n-2} \cdots x_n^{\lambda_n + n-n})}{\sum_{w \in S_n} \det(w)\, w(x_1^{n-1} x_2^{n-2} \cdots x_n^{n-n})}$$

where $w x_i = x_{w(i)}$ and $w(fg) = w(f)w(g)$.