<u>Representation Theory</u> <u>Friday</u> 09.09.2011. ①

<u>Cohomology of flag varieties</u>

$$G = GL_n(\mathbb{C})$$
$$\cup\!|$$
$$B = \left\{ \begin{pmatrix} * & \ddots & * \\ & & \\ 0 & & * \end{pmatrix} \right\}$$

The <u>flag variety</u> is G/B

Let $J \subseteq \{1, 2, \ldots, n-1\}$. The <u>parabolic subgroup</u> P_J is the subgroup of block upper triangular matrices with blocks ending according to J. For example If $J = \{3, 5, 6, 10\}$ in $\{1, 2, \ldots, 12\}$ then

$$P_{35610} = \left\{ \begin{pmatrix} \begin{matrix} * & * & * \\ * & * & * \\ * & * & * \end{matrix} & & & & * \\ & \begin{matrix} * & * \\ * & * \end{matrix} & & & \\ & & * & & \\ & & & \begin{matrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{matrix} & \\ 0 & & & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & \begin{matrix} * & * \\ * & * \end{matrix} \end{pmatrix} \right\} \quad \text{in } GL_{12}(\mathbb{C})$$

G/P_J are the <u>partial flag varieties</u>

For $G = GL_3(\mathbb{C})$

$$\begin{tikzcd} & G/B = G/P_{12} \arrow[dl] \arrow[dr] & \\ G/P_2 \arrow[dr] & & G/P_1 \arrow[dl] \\ & G/G = G/P_\emptyset & \end{tikzcd}$$

with $P_1 = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\}$

$P_2 = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \right\}$

If $J = \{1\}$ then $G/P_J = \mathbb{P}^{n-1}$, <u>projective space</u>.

If $J = \{k\}$ then $G/P_J = Gr_{k,n}$, the <u>Grassmannian</u> of "k-planes in $\mathbb{C}$".

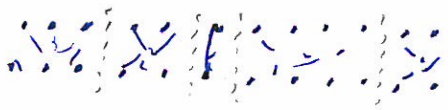
The <u>symmetric group</u> $W_0 = S_n$ is generated by

$$s_i = \overset{1\ 2\ \cdots\ \ i\ \ i+1\ \cdots\ n}{|\,|\,|\cdots|\ X\ |\cdots|}, \quad \text{with } i = 1, 2, \ldots, n-1.$$

Let W_J be the subgroup generated by $\{s_i \mid i \notin J\}$.

If $J = \{3, 5, 6, 10\}$ then, in S_{12}

$$W_J = \langle s_1, s_2, s_7, s_8, s_9, s_{11} \rangle = S_3 \times S_2 \times S_1 \times S_4 \times S_2$$

$= \{$  $\mid$ edges don't cross the dotted lines $\}$

Let W^J be a set of coset representatives of W/W_J.

The <u>Bruhat</u> <u>decompositions</u> are

$$G = \bigsqcup_{w \in W_0} BwB \quad \text{and} \quad G = \bigsqcup_{u \in W^J} BuP_J \qquad \text{(row reduction)}$$

Then

$$X_w = \overline{BwB} \text{ in } G/B$$
$$X_u = \overline{BuP_J} \text{ in } G/P_J$$

are the <u>Schubert</u> <u>varieties</u>

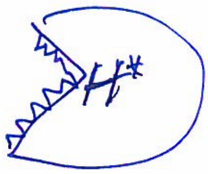
where $\overline{A}$ is the closure of A (small closed set containing A).

<u>Example</u> $G = GL_3(\mathbb{C})$, $W_0 = S_3 = \langle s_1, s_2 \mid s_i^2 = 1,\ s_1 s_2 s_1 = s_2 s_1 s_2 \rangle$

$$\begin{tikzcd}
 & X_{s_1 s_2 s_1} = G/B & \\
X_{s_1 s_2} \arrow[ur, hook] & & X_{s_2 s_1} \arrow[ul, hook'] \\
X_{s_1} \arrow[u, hook] \arrow[urr, hook] & & X_{s_2} \arrow[u, hook'] \arrow[ull, hook'] \\
 & X_1 \arrow[ul, hook] \arrow[ur, hook'] &
\end{tikzcd}$$

in G/B.

Cohomology

spaces $\longrightarrow$ 

$\longrightarrow$ commutative graded rings

such that

If $f: X \to Y$ is a morphism between spaces then there are "morphisms"

$$f^*: H^*(Y) \longrightarrow H^*(X) \quad \text{and} \quad f_*: H^*(X) \longrightarrow H^*(Y)$$

So we have for $G = GL_3(\mathbb{C})$

$$\begin{tikzcd} & H^*(G/B) & \\ H^*(G/P_2) \arrow[ur] & & H^*(G/P_1) \arrow[ul] \\ & H^*(G/G) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

and

$$\begin{tikzcd} & H^*(G/B) \arrow[dl] \arrow[dr] & \\ H^*(G/P_2) \arrow[dr] & & H^*(G/P_1) \arrow[dl] \\ & H^*(G/G) = H^*(pt) & \end{tikzcd}$$

and

$$\begin{tikzcd} & H^*(G/B) \arrow[dl] \arrow[dr] & \\ H^*(X_{s_1s_2}) \arrow[d] \arrow[drr] & & H^*(X_{s_2s_1}) \arrow[d] \arrow[dll] \\ H^*(X_{s_1}) \arrow[dr] & & H^*(X_{s_2}) \arrow[dl] \\ & H^*(X_1) & \end{tikzcd}$$

and

$$\begin{tikzcd} & H^*(G/B) & \\ H^*(X_{s_1s_2}) \arrow[ur] & & H^*(X_{s_2s_1}) \arrow[ul] \\ H^*(X_{s_1}) \arrow[u] \arrow[urr] & & H^*(X_{s_2}) \arrow[ull] \\ & H^*(X_1) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Borel's Theorem

Theorem

$$H_T^*(G/B) = \mathbb{C}[y_1,\ldots,y_n] \otimes_{\mathbb{C}[x_1,\ldots,x_n]^{W_0}} \mathbb{C}[x_1,\ldots,x_n]$$

Recall that

$$\mathbb{C}[x_1,\ldots,x_n]^{W_0} = \left\{ f \in \mathbb{C}[x_1,\ldots,x_n] \;\middle|\; \begin{array}{l} f(x_1,\ldots,x_n) = f(x_{w(1)},\ldots,x_{w(n)}) \\ \text{for } w \in S_n \end{array} \right\}$$

$$= \mathbb{C}[e_1,\ldots,e_n] = \mathrm{span}\{ e_1^{\lambda_1}\cdots e_n^{\lambda_n} \mid \lambda_1,\ldots,\lambda_n \in \mathbb{Z}_{\geq 0}\}$$

where $e_\ell = \sum_{1\leq i_1<\cdots<i_\ell\leq n} x_{i_1}\cdots x_{i_\ell}$ $\left(\begin{array}{l}\text{so } e_1 = x_1+x_2+\cdots+x_n, \\ e_n = x_1x_2\cdots x_n, \\ e_2 = x_1x_2+x_1x_3+\cdots+x_2x_3+x_2x_4+\cdots+x_{n-1}x_n\end{array}\right)$

Thus

$$H_T^*(G/B) = \frac{\mathbb{C}[x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_n]}{\langle f(x_1,\ldots,x_n) = f(y_1,\ldots,y_n) \text{ if } f\in\mathbb{C}[x_1,\ldots,x_n]^{W_0}\rangle}$$

$$= \frac{\mathbb{C}[x_1,\ldots,x_n,y_1,y_2,\ldots,y_n]}{\langle e_\ell(x_1,\ldots,x_n) = e_\ell(y_1,\ldots,y_n) \text{ for } \ell=1,2,\ldots,n\rangle}$$

and

$$H_T^*(G/P_J) = \mathbb{C}[y_1,\ldots,y_n] \otimes_{\mathbb{C}[x_1,\ldots,x_n]^{W_0}} \mathbb{C}[x_1,\ldots,x_n]^{W_J}$$

where $\mathbb{C}[x_1,\ldots,x_n]^{W_J} = \left\{ f\in\mathbb{C}[x_1,\ldots,x_n] \;\middle|\; \begin{array}{l} f(x_1,\ldots,x_n) = f(x_{w(1)},\ldots,x_{w(n)}) \\ \text{for } w\in W_J\end{array}\right\}$

So all questions about $H_T^*(G/B)$ are converted to questions about polynomials and symmetric polynomials.

The map $T_J : H_T^*(G/B) \longrightarrow H_T^*(G/P_J)$ is given by

$$f \longmapsto T_J(f)$$

$$T_J(f) = \Big(\sum_{w \in W_J} w\Big) \frac{1}{D_J} f\,, \quad \text{for } f \in \mathbb{C}[y_1, \ldots, y_n, x_1, \ldots, x_n]$$

where $w x_i = x_{w(i)}$, $w y_i = y_i$ and $D_J = \prod_{\substack{k<l \\ \text{in same block} \\ \text{of } J^c}} (x_k - x_l)$

where k, l are in the same block of J^c

if there does not exist $j \in J$ with $k \le j < l$.

For example, if $J = \{3, 5, 6, 10\}$ in $\{1, 2, \ldots, 12\}$ then

$$\begin{aligned} D_J = {} & (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \cdot (x_4 - x_5) \\ & \cdot (x_6 - x_7)(x_6 - x_8)(x_6 - x_9)(x_6 - x_{10})(x_7 - x_8)(x_7 - x_9)(x_7 - x_{10}) \\ & \cdot (x_8 - x_9)(x_8 - x_{10})(x_9 - x_{10}) \\ & \cdot (x_{11} - x_{12}) \end{aligned}$$