Representation Theory lecture 26.08.2011.

The group $GL_n(\mathbb{F})$.

$$GL_n(\mathbb{F}) = \left\{ \begin{array}{l} n\times n \text{ invertible matrices with} \\ \text{entries in } \mathbb{F} \end{array} \right\}$$

$$= \{ g\in M_n(\mathbb{F}) \mid g^{-1}\in M_n(\mathbb{F}) \}$$

$$= \{ g\in M_n(\mathbb{F}) \mid \det(g) \neq 0 \}.$$

The basic data for $GL_n(\mathbb{F})$ is

$$\mathfrak{h}_{\mathbb{Z}} = \sum_{i=1}^{n} \mathbb{Z}\varepsilon_i, \quad \text{with} \quad W_0 = S_n$$

acting by permuting $\varepsilon_1,\dots,\varepsilon_n$. The reflecting hyperplanes are

$$\mathfrak{h}^{\varepsilon_i-\varepsilon_j} = \{ \lambda\in\mathfrak{h}_{\mathbb{R}} \mid \langle \lambda, \varepsilon_i-\varepsilon_j\rangle = 0 \} \quad \text{for } 1\le i<j\le n.$$

where

$$\mathfrak{h}_{\mathbb{R}} = \mathbb{R}\otimes_{\mathbb{Z}}\mathfrak{h}_{\mathbb{Z}} \quad \text{and} \quad \langle \varepsilon_i, \varepsilon_j\rangle = \delta_{ij}.$$

Let

$$s_{ij} = 1 - E_{ii} - E_{jj} + E_{ij} + E_{ji} \quad \text{and} \quad x_{ij}(f) = 1 + f e_{ij}, \text{ for } f\in\mathbb{F}.$$

$$h_\lambda(g) = \mathrm{diag}(g^{\lambda_1},\dots,g^{\lambda_n}), \text{ for } g\in\mathbb{F}^\times.$$

<u>Theorem</u> $GL_n(\mathbb{F})$ is generated by

$$s_{ij},\ x_{ij}(f),\ x_{ji}(f) \text{ and } h_\lambda(g)$$

with relations

$$x_{ij}(f_1)\,x_{ij}(f_2) = x_{ij}(f_1+f_2)$$
$$h_\lambda(g_1)\,h_\lambda(g_2) = h_\lambda(g_1g_2)$$
$$h_\lambda(g_1)\,h_\mu(g_1) = h_{\lambda+\mu}(g_1)$$

$$x_{ij}(f_1)\,x_{kl}(f_2) = x_{kl}(f_2)\,x_{ij}(f_1) \quad \text{if } i\neq l,\ j\neq k,$$
$$x_{ij}(f_1)\,x_{jl}(f_2) = x_{jl}(f_2)\,x_{ij}(f_1)\,x_{il}(f_1f_2), \quad \text{if } i\neq l$$
$$x_{ij}(f_1)\,x_{ki}(f_2) = x_{ki}(f_2)\,x_{ij}(f_1)\,x_{kj}(-f_1f_2), \quad \text{if } j\neq k$$

$$x_{ij}(g)\,x_{ji}(-g^{-1})\,x_{ij}(g) = h_{\varepsilon_j}(-1)\,h_{\varepsilon_i-\varepsilon_j}(g)\,s_{ij}$$
$$h_\lambda(g)\,x_{ij}(f)\,h_\lambda(g)^{-1} = x_{ij}(f g^{\langle\lambda,\varepsilon_i-\varepsilon_j\rangle})$$
$$w\,x_{ij}(f)\,w^{-1} = x_{w(i)w(j)}(f), \quad \text{for } w\in S_n.$$

Proof Row reduction.

Multiplication by $x_{ij}(f)$ on the left are called row operations.

Using these and multiplications by $h_{\varepsilon_i}(g)$ there is a matrix in row echelon form

$$R = \begin{pmatrix} 1 & * & * & * & 0 & * & * & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = P\cdot A$$

for an $A\in M_{n\times m}(\mathbb{F})$, $P\in GL_n(\mathbb{F})$

If $A\in GL_n(\mathbb{F})$ then $R = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1\end{pmatrix}$, the identity.