Presenting the affine Hecke algebra: Iwahori and Bernstein Presentations and the Path model (CBMS Lecture 1)

Initial data

The initial data is

a finite $\mathbb{Z}$-reflection group. $(W_0, \mathfrak{h}_{\mathbb{Z}})$

This means

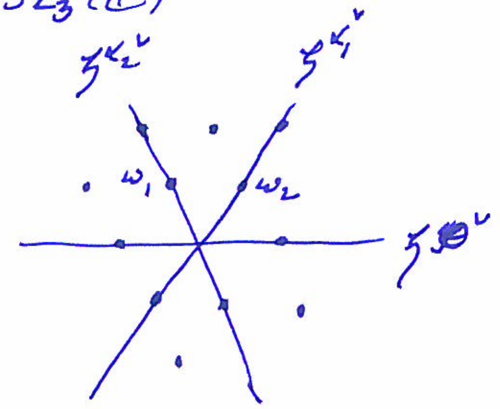
$\mathfrak{h}_{\mathbb{Z}}$ is a free $\mathbb{Z}$-module (has $\mathbb{Z}$-basis $\{\omega_1, \ldots, \omega_\ell\}$)

W_0 is a finite subgroup of $GL(\mathfrak{h}_{\mathbb{Z}})$ generated by reflections.

A <u>reflection</u> is a matrix conjugate (in $GL(\mathfrak{h}_{\mathbb{C}})$) to

$$\begin{pmatrix} \xi & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \quad \text{with } \xi \neq 1.$$

<u>Our favorite example</u>: Type $SL_3(\mathbb{C})$



$\mathfrak{h}_{\mathbb{Z}} = \text{span}\{\omega_1, \omega_2\}$

W_0 = dihedral group of order 6 generated by s_1, s_2

$W_0 = \{1, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}$

contains 3 reflections, s_1, s_2 and $s_1 s_2 s_1 = s_\theta$,

the reflections in $\mathfrak{h}^{\alpha_1^\vee}, \mathfrak{h}^{\alpha_2^\vee}, \mathfrak{h}^{\theta^\vee}$, respectively.

<u>The affine Weyl group</u> (semidirect product presentation)

$$W = \{t_w X^\lambda \mid w \in W_0, \lambda \in \mathfrak{h}_{\mathbb{Z}}\}$$

with

$$t_u t_v = t_{uv}, \quad X^\lambda X^\mu = X^{\lambda+\mu} \text{ and } t_w X^\lambda = X^{w\lambda} t_w$$

for $u, v, w \in W_0$ and $\lambda, \mu \in \mathfrak{h}_{\mathbb{Z}}$. Then W acts on

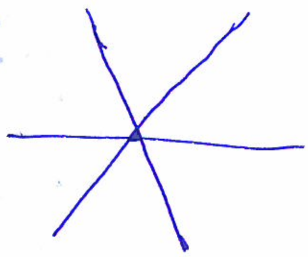
$$\mathfrak{h} = \mathfrak{h}_{\mathbb{R}} \oplus \mathbb{R}\Lambda_0 \qquad \left(\mathfrak{h}_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} \mathfrak{h}_{\mathbb{Z}} = \mathbb{R}\text{-span}\{\omega_1, \ldots, \omega_\ell\}\right)$$

by

$$t_w X^\lambda(\mu + m\Lambda_0) = \left(\begin{array}{ccc|c} & t_w & & \lambda \\ \hline 0 & \cdots & 0 & 1 \end{array}\right)\left(\begin{array}{c} \mu \\ \hline m \end{array}\right) = \left(\begin{array}{c} t_w\mu + m\lambda \\ \hline m \end{array}\right) = t_w\mu + m\lambda + m\Lambda_0$$

In our <u>$SL_3(\mathbb{C})$-example</u>

$\mathfrak{h}_0 = \mathfrak{h}_{\mathbb{R}} \oplus 0\Lambda_0$

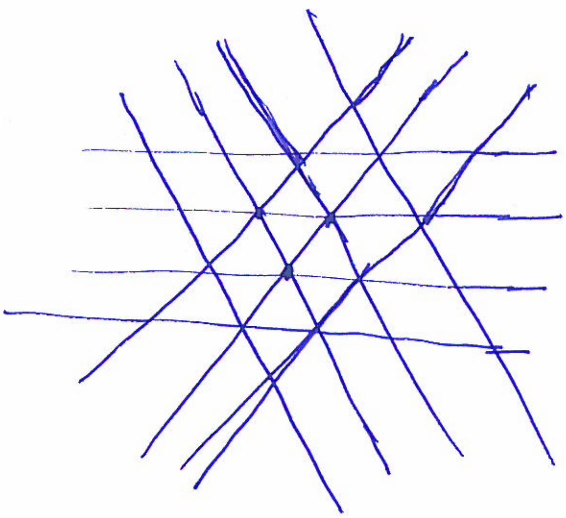


X^λ acts trivially

$t_w \mu = w\mu$

$X^\lambda \mu = \mu$

$\mathfrak{h}_1 = \mathfrak{h}_{\mathbb{R}} + \Lambda_0$



Coxeter generators

C_m are fundamental regions for W acting on $\mathfrak{h}_m$ such that

$$\cancel{\cdots \supseteq \bar C_0 \supseteq \bar C_1 \supseteq \bar C_0} \quad \bar C_0 \supseteq \cdots \quad \supseteq \bar C_2 \supseteq \bar C_1 \supseteq 0.$$

$\mathfrak{h}^{\alpha_1^\vee}, \dots, \mathfrak{h}^{\alpha_l^\vee}$ are the walls of C_0

$\mathfrak{h}^{\alpha_0^\vee}, \dots, \mathfrak{h}^{\alpha_l^\vee}$ are the walls of C_1,

$s_0, \dots, s_l$ the corresponding reflections.

$$\Omega = \{ g \in W \mid g C_1 = C_1 \}.$$

__Theorem__ W is presented by generators $s_0, s_1, \dots, s_n$ and Ω such that Ω is a subgroup,

$s_i^2 = 1$, for $i = 0, 1, \dots, l$,

$\underbrace{s_i s_j s_i \cdots}_{m_{ij}\text{ factors}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}\text{ factors}}$ for $i \neq j$, where $\frac{\pi}{m_{ij}} = \mathfrak{h}^{\alpha_i^\vee} \angle\, \mathfrak{h}^{\alpha_j^\vee}$

$g s_i g^{-1} = s_{g(i)}$, where $g \mathfrak{h}^{\alpha_i^\vee} = \mathfrak{h}^{\alpha_{g(i)}^\vee}$.

The __Dynkin__, or __Coxeter__, __diagram__ of W is the graph with vertices $\alpha_0^\vee, \dots, \alpha_l^\vee$

and edges $\alpha_i^\vee \overset{m_{ij}}{\text{———}} \alpha_j^\vee$

(the graph of the "1-skeleton of C_1").

<u>The affine Hecke algebra</u> (Bernstein presentation)

H is generated by $T_1, \ldots, T_\ell$ and X^λ, $\lambda \in \mathfrak{h}_{\mathbb{Z}}$ with

$$T_i^2 = 1 \quad \text{for } i = 1, 2, \ldots, \ell, \qquad X^\lambda X^\mu = X^{\lambda+\mu},$$

$$\underbrace{T_i T_j T_i \cdots}_{m_{ij} \text{ factors}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij} \text{ factors}} \quad \text{for } i \neq j$$

$$T_i X^\lambda = X^{s_i\lambda} + (q - q^{-1}) \frac{X^\lambda - X^{s_i\lambda}}{1 - X^{-\alpha_i}}$$

where $s_i\lambda = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$. determines $\alpha_i \in \mathfrak{h}_{\mathbb{Z}}$.

<u>The affine Hecke algebra</u> (Coxeterish presentation)

H is presented by generators $T_0, T_1, \ldots, T_\ell$ and Ω with

$$T_i^2 = (q - q^{-1}) T_i + 1, \quad \text{for } i = 0, 1, \ldots, \ell,$$

$$\underbrace{T_i T_j T_i \cdots}_{m_{ij} \text{ factors}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij} \text{ factors}} \quad \text{for } i \neq j$$

$$g h = gh \quad \text{and} \quad g T_i g^{-1} = T_{g(i)}$$

for $g, h \in \Omega$.

Bases of H

Let $w \in W$. A reduced word for w,

$$w = g s_{i_1} \cdots s_{i_L}, \quad g \in \Omega, \quad i_1, \dots, i_L \in \{0, 1, \dots, \ell\}$$

is a minimal length sequence

$$\vec{w} = (g, \ C_1 \xrightarrow{\mathfrak{h}^{\beta_1}} s_{i_1} C_1 \xrightarrow{\mathfrak{h}^{\beta_2}} s_{i_1} s_{i_2} C_1 \to \cdots \xrightarrow{\mathfrak{h}^{\beta_L}} s_{i_1} \cdots s_{i_L} C_1)$$

The periodic orientation is

(a) If $0 \in \mathfrak{h}^{\alpha}$ then C_0 is on the positive side of $\mathfrak{h}^{\alpha}$

(b) Parallel hyperplanes have parallel orientation.

For a reduced word $w = g s_{i_1} \cdots s_{i_L}$ define

$$T_w = g T_{i_1} \cdots T_{i_L} \quad \text{and} \quad X^w = g T_{i_1}^{\epsilon_1} \cdots T_{i_L}^{\epsilon_L}$$

where

$$\epsilon_j = \begin{cases} +1, & \text{if the } j^{\text{th}} \text{ step of } \vec{w} \text{ is } \overset{-\ \mathfrak{h}^{\beta_j}\ +}{\longrightarrow} \\ -1, & \text{if the } j^{\text{th}} \text{ step of } \vec{w} \text{ is } \overset{-\ \mathfrak{h}^{\beta_j}\ +}{\longleftarrow} \end{cases}$$

Then

$$\{T_w \mid w \in W\}, \qquad \{X^w \mid w \in W\}$$

$$\{T_v X^{\lambda} \mid v \in W_0, \lambda \in \mathfrak{h}_{\mathbb{Z}}\}, \qquad \{X^{\lambda} T_w \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}, w \in W_0\}$$

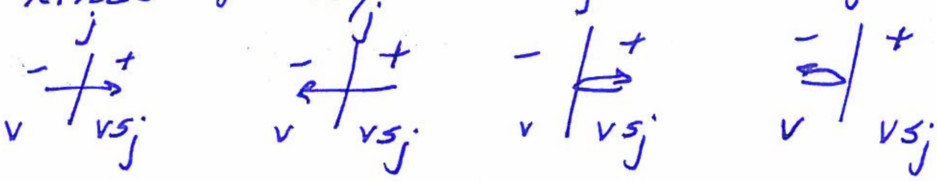
are bases of H.

Note:

$$X^{\lambda} = X^{t_{\lambda}} \quad \text{and} \quad X^{w} \overset{?}{=} T_v X^{\mu} \text{ if } w = v t_{\mu}$$

The path model (the algebra of paths).

For kinds of steps

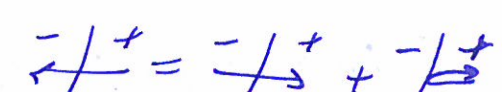


An *alcove walk* is a sequence of steps such that

(a) the tail of the first step is in C_1

(b) at every step, the head of each arrow is in the same alcove as the tail of the next.

Use the relations

 and 

to straighten any sequence of steps to a linear combination of alcove walks.

Theorem Fix $\lambda \in \mathfrak{h}_{\mathbb{Z}}$ and $w \in W_0$.

Fix a minimal length walk $c_{i_1}\cdots c_{i_r}$ from C_1 to wC_1 and a minimal length walk $c_{j_1}^{\epsilon_1}\cdots c_{j_s}^{\epsilon_s}$ from C_1 to $\lambda + C_1$

Then

$$T_{w^{-1}}X^{\lambda} = \sum_{p} (-1)^{\#\text{neg folds of } p} (q-q^{-1})^{\#\text{of folds of } p} X^{wt(p)} T^{-1}_{\varphi(p)^{-1}}$$

where the sum is over all alcove walks

$$c_{i_1}\cdots c_{i_r} p_{j_1}\cdots p_{j_s} \quad \text{where } p_{j_k} \text{ is } c_{j_k}^{+} \text{ or } c_{j_k}^{-} \text{ or } f_{j_k}^{\epsilon_k}$$

and

$$end(p) = wt(p) + \varphi(p) C_1$$

is the ending alcove of p.