Let S be a set. Let A be an algebra

A <u>free</u> <u>A-module</u> <u>on S</u> is a pair (A^S, ι) where

A^S is an A-module

$\iota : S \to A^S$ is a function

such that, if M is an A-module with a function $j : S \to M$ then there exists a unique morphism $\tilde{j} : A^S \to M$ such that $\tilde{j} \circ \iota = j$.

$$\begin{tikzcd} S \arrow[r, "\iota"] \arrow[dr, "j"'] & A^S \arrow[d, dashed, "\tilde{j}"] \\ & M \end{tikzcd}$$

This is a definition by a <u>universal</u> <u>property</u>.

A morphism in the category of sets is a <u>function</u>.

A morphism in the category of categories is a <u>functor</u>.

The <u>forgetful</u> <u>functor</u> is

$$\begin{array}{rcl} F : A\text{-modules} & \longrightarrow & \text{Sets} \\ M & \longmapsto & M \end{array}$$

The <u>free module</u> <u>functor</u> is

$$\begin{array}{rcl} \text{Sets} & \longrightarrow & A\text{-modules} \\ S & \longmapsto & A^S \end{array}$$

The universal property for free modules tells us

$$\mathrm{Hom}_A(A^S, M) \simeq \mathrm{Hom}_{Sets}(S, M)$$
$$\tilde{f} \longleftarrow f$$

The free module functor is the <u>left adjoint</u> to the forgetful functor.

Let M be an A-module.

A <u>presentation</u> of M is an exact sequence

$$A^T \to A^S \to M \to 0$$

where A^T and A^S are free modules.

The term <u>exact</u> <u>sequence</u> means that

(a) $\mathrm{im}\,\rho = \ker \gamma$

(b) $\mathrm{im}\,\gamma = \ker 0$.

Let X be a group

A <u>presentation</u> of X is an exact sequence

$$R \to G \to X \to \{1\}$$

where R and G are free groups.

<u>Example</u>

(1) $\mathbb{Z}/m\mathbb{Z}$ is generated by 1 with relation $\underbrace{1+1+\cdots+1}_{m \text{ times}} = 0$.

The <u>cyclic group of order m</u>

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C_m is generated by g with relation $g^m = 1$.

Alternatively:

$\mathbb{Z}/m\mathbb{Z}$ is presented by

$$\begin{array}{ccccccc} \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}/m\mathbb{Z} & \longrightarrow & 0 \\ & & 1 & \longmapsto & 1 & & \\ 1 & \longmapsto & m & & & & \end{array}$$

and C_m is presented by

$$\begin{array}{ccccccc} G^{\{r\}} & \longrightarrow & G^{\{g\}} & \longrightarrow & C_m & \longrightarrow & \{1\} \\ & & g & \longmapsto & g & & \\ r & \longmapsto & g^m & & & & \end{array}$$

where G^S denotes the free group on the set S.

<u>Example 2</u>

The <u>dihedral group of order $2m$</u>, $G_{m,m,2}$ is generated by x and y with relations

$x^2 = 1$, $y^m = 1$ and $xy = y^{-1}x$

Alternatively:

$$\begin{array}{ccccccc} F_3 & \longrightarrow & F_2 & \longrightarrow & G_{m,m,2} & \longrightarrow & \{1\} \\ r_3 & \longmapsto & g_1 g_2 g_1^{-1} g_2 & & & & \\ & & g_1 & \longmapsto & x & & \\ & & g_2 & \longmapsto & y & & \\ r_1 & \longmapsto & g_1^2 & & & & \\ r_2 & \longmapsto & g_2^m & & & & \end{array}$$

where F_k denotes a free group on a set with k elements

Let $\mathbb{F}$ be a field

An <u>algebra</u> is a vector space A with a linear transformation

$$\begin{aligned} A\otimes A &\longrightarrow A \\ a\otimes b &\longmapsto ab \end{aligned}$$

such that

(a) If $a_1, a_2, a_3 \in A$ then $(a_1a_2)a_3 = a_1(a_2a_3)$,

(b) There exists $1 \in A$ such that
if $a \in A$ then $1\cdot a = a$ and $a\cdot 1 = a$.

Let A be an algebra. An <u>A-module</u> is a vector space M with a linear transformation

$$\begin{aligned} A\otimes M &\longrightarrow M \\ a\otimes m &\longmapsto am \end{aligned} \qquad \text{such that}$$

(a) If $a_1, a_2 \in A$ and $m \in M$ then $a_1(a_2m) = (a_1a_2)m$,

(b) If $m \in M$ then $1\cdot m = m$.

Let A be an algebra. Let M, N be A-modules. A <u>morphism from M to N</u> is a linear transformation $f : M \to N$ such that

if $a \in A$ and $m \in M$ then $f(am) = af(m)$.

Let A and B be algebras. A <u>morphism from A to B</u> is a linear transformation $f: A \to B$ such that

(a) if $a_1, a_2 \in A$ then $f(a_1 a_2) = f(a_1) f(a_2)$, and

(b) $f(1) = 1$.

Let $f: A \to B$ be a morphism of algebras.
Let N be a B-module.
Define $f^*(N)$ to be the A-module with

- vector space N and
- A-action given by $an = f(a)n$.

Then f^*, <u>restriction along f</u>, is a functor

$$f^*: B\text{-modules} \longrightarrow A\text{-modules}.$$

If A is a subalgebra of B, $A \hookrightarrow B$,
the functor f^* is called <u>restriction</u> and denoted Res_A^B,

$$\begin{aligned} \mathrm{Res}_A^B : B\text{-modules} &\longrightarrow A\text{-modules} \\ N &\longmapsto \mathrm{Res}_A^B(N) \end{aligned}$$

<u>Induction</u> is the left adjoint to restriction

$$\begin{aligned} \mathrm{Ind}_A^B : A\text{-modules} &\longrightarrow B\text{-modules} \\ M &\longmapsto \mathrm{Ind}_A^B(M) \end{aligned}$$

determined by

$$\mathrm{Hom}_B(\mathrm{Ind}_A^B(M), N) \simeq \mathrm{Hom}_A(M, \mathrm{Res}_A^B(N))$$

$$\tilde{\varphi} \longleftarrow\!\shortmid \varphi$$

Alternatively: $\mathrm{Ind}_A^B(M)$ is a pair $(\mathrm{Ind}_A^B(M), \iota)$ where

$\mathrm{Ind}_A^B(M)$ is a B-module with

a morphism $\iota: M \to \mathrm{Ind}_A^B(M)$

such that if N is a B-module with an

morphism $\varphi: M \to N$

then there exists a unique $\tilde{\varphi}: \mathrm{Ind}_A^B(M) \to N$

such that $\tilde{\varphi} \circ \iota = \varphi$.

$$\begin{tikzcd} M \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & \mathrm{Ind}_A^B(M) \arrow[d, dashed, "\tilde{\varphi}"] \\ & N \end{tikzcd}$$