Categories

Examples:
(0) The category of sets.
(1) The category of groups
(2) The category of abelian groups
(3) The category of rings
(4) The category of R-modules
(5) The category of vector spaces
(6) The category of topological spaces
(7) The category of topological groups
(8) The category of manifolds
(9) The category of complex manifolds.
(∞) The category of categories

A category $\mathcal{C}$ has

Objects and morphisms.

$Hom_{\mathcal{C}}(X,Y)$ is the set of morphisms from X to Y

Let $\mathbb{F}$ be a field.

An <u>algebra</u> is a vector space A with a function

$$A \times A \longrightarrow A$$
$$(a,b) \longmapsto ab$$

such that

(a) If $a_1, a_2, a_3 \in A$ then $(a_1 a_2) a_3 = a_1 (a_2 a_3)$

(b) There exists $1 \in A$ such that if $a \in A$ then $1 \cdot a = a \cdot 1 = a$

(c) If $a_1, a_2, a_3 \in A$ then

$$(a_1 + a_2) a_3 = a_1 a_3 + a_2 a_3 \quad \text{and} \quad a_1(a_2 + a_3) = a_1 a_2 + a_1 a_3$$

Let A be an algebra.

An <u>A-module</u> is a vector space M with a function

$$\begin{aligned} A\otimes M &\to M \\ a\otimes m &\mapsto am \end{aligned} \qquad \text{such that}$$

(a) If $a_1, a_2 \in A$ and $m \in M$ then $a_1(a_2 m) = (a_1 a_2)m$,

(b) If $m \in M$ then $1m = m$,

(c) If $c_1, c_2 \in \mathbb{F}$ and $m_1, m_2 \in M$ and $a_1, a_2 \in A$ then

$$a(c_1 m_1 + c_2 m_2) = c_1(am_1) + c_2(am_2).$$

<u>Representation theory</u> is the study of the category of A-modules.

<u>Examples</u>: (0) $\mathbb{F}$ is an $\mathbb{F}$-algebra

(1) $M_n(\mathbb{F})$ is an $\mathbb{F}$-algebra

(2) $\mathbb{F}[t]$ is an $\mathbb{F}$-algebra

(3) $\mathbb{F}[t_1, t_2, \ldots, t_n]$ is an $\mathbb{F}$-algebra.

(4) $\mathbb{C}$ is the $\mathbb{R}$-algebra generated by 1 and i with relation

$$i^2 = -1.$$

(this already forces
$(a_1 + ia_2)(b_1 + ib_2) = a_1 b_1 - a_2 b_2 + i(a_1 b_2 + a_2 b_1)$).

(5) The Weyl algebra is the $\mathbb{C}$-algebra generated by p, q with relation

$$pq - qp = i\hbar$$

(an important case is $\hbar = \frac{1}{i}$ because, as operators on $\mathbb{C}[x]$,

$$x\frac{\partial}{\partial x} - \frac{\partial}{\partial x}x = 1.\text{)}$$

$$\frac{\partial}{\partial x}xf = x\frac{\partial f}{\partial x} + 1 \cdot f, \text{ so that } \left(\frac{\partial}{\partial x}x\right) \cdot f = \left(x\frac{\partial}{\partial x} + 1\right)f.$$

Let V be a vector space with basis $x_1, \dots, x_n$ and ~~...~~ let $\langle , \rangle : V \otimes V \to \mathbb{R}$ be an inner product on V. The Weyl algebra is the algebra A generated by $x_1, \dots, x_n$ and $y_1, \dots, y_n$ with relations

$$x_i x_j = x_j x_i, \quad y_i y_j = y_j y_i \quad \text{and}$$

$$x_i y_j - y_j x_i = \langle x_i, x_j \rangle \hbar i$$

A useful A-module, for the case $\hbar = \frac{1}{i}$ and $\langle x_i, x_j \rangle = \delta_{ij}$ is

$\mathbb{C}[x_1, \dots, x_n]$ with x_i acting by multiplication by x_i and y_i acting by $\frac{\partial}{\partial x_i}$.

Let A be an algebra. Let M, N be A-modules.
A morphism from M to N is a linear transformation $f: M \to N$ such that

if $a \in A$ and $m \in M$ then $f(am) = af(m)$.

This specifies $\mathrm{Hom}(M, N)$ in the category of A-modules