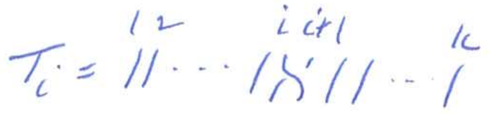
The *braid group on k strands* $\mathcal{B}_k$ is generated by

$$T_i =$$  $$, \quad 1 \le i \le k-1$$

with relations

$$T_iT_j = T_jT_i \text{ if } j \neq i \pm 1, \text{ and } T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}.$$

Let

$$y^{\varepsilon_i^\vee} =$$  $$, \quad \text{for } i = 1, 2, \ldots, k,$$

$$= T_{i-1}T_{i-2} \cdots T_2T_1^2T_2 \cdots T_{i-1}.$$

Then

$$y^{\varepsilon_i^\vee}y^{\varepsilon_j^\vee} = y^{\varepsilon_j^\vee}y^{\varepsilon_i^\vee} \text{ and } y^{\varepsilon_1^\vee} \cdots y^{\varepsilon_n^\vee} = T_{w_0}^2 \in Z(\mathcal{B}_k)$$

where

$$T_{w_0} =$$  and $Z(\mathcal{B}_k)$ is the center of $\mathcal{B}_k$.

$$\mathcal{B}_k \hookrightarrow \mathcal{B}_{k+1}$$

so that

$$\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \mathcal{B}_3 \subseteq \mathcal{B}_4 \subseteq \cdots$$

If M is a finite dimensional simple $\mathcal{B}_k$-module then $z \in Z(\mathcal{B}_k)$ acts on M by a constant. This follows from Schur's lemma, which says

$$\mathrm{End}_{\mathcal{B}_k}(M) = \mathbb{C} \cdot \mathrm{id}_M, \text{ if } M \text{ is simple}.$$

This statement means:

$$\left\{\begin{array}{c}\text{Irreducible}\\ H_k\text{-modules}\end{array}\right\} \xleftrightarrow{1\text{-}1} \hat{H}_k$$

$$H_k^\lambda \longleftarrow\!\!\shortmid \lambda$$

and

$$\mathrm{Res}^{H_k}_{H_{k-1}}(H_k^\lambda) = \bigoplus_{\substack{\mu \subseteq \lambda \\ \lambda/\mu = \square}} H_{k-1}^\mu$$

Induction is the adjoint functor to restriction:

$$\mathrm{Hom}_{H_k}(\mathrm{Ind}^{H_k}_{H_{k-1}}(H_{k-1}^\mu), H_k^\lambda) \simeq \mathrm{Hom}_{H_{k-1}}(H_{k-1}^\mu, \mathrm{Res}^{H_k}_{H_{k-1}}(H_k^\lambda))$$

and

$$\mathrm{Hom}_{H_k}(H_k^\lambda, H_k^\nu) = \begin{cases} 0, & \text{if } \lambda \neq \nu \\ \mathbb{C}\cdot \mathrm{Id}, & \text{if } \lambda = \nu \end{cases}$$

so that

$$\mathrm{Ind}^{H_k}_{H_{k-1}}(H_{k-1}^\mu) = \bigoplus_{\substack{\lambda \supseteq \mu \\ \lambda/\mu = \square}} H_k^\lambda .$$

We get that

$$\dim(H_k^\lambda) = \mathrm{Card}\left(\left\{\begin{array}{c}\text{paths } \emptyset \to \cdots \to \lambda \\ \text{in } \hat{H}\end{array}\right\}\right)$$

<u>Example</u> As vector spaces

$$H_4^{\tiny\begin{smallmatrix}\square\square\square\\\square\end{smallmatrix}} = H_3^{\tiny\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} \oplus H_3^{\square\square\square} = H_2^{\tiny\begin{smallmatrix}\square\\\square\end{smallmatrix}} \oplus H_2^{\square\square} \oplus H_2^{\square\square}$$

$$= H_1^\square \oplus H_1^\square \oplus H_1^\square$$

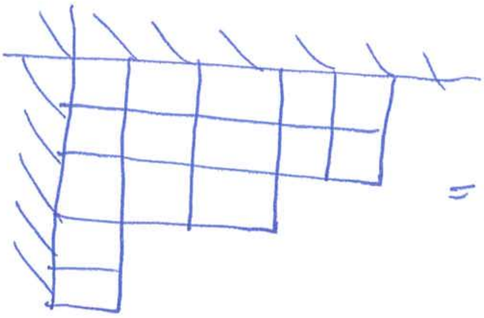
and $H_1 \simeq M_1(\mathbb{C})$ has a unique simple module of dimension 1.

Let $q \in \mathbb{C}^\times$.

The Iwahori-Hecke algebra $H_k(q)$ is the quotient of $\mathbb{C}B_k$ by the relations

$$T_i^2 = (q-q^{-1})T_i + 1, \quad \text{for } i=1,\ldots,k-1.$$

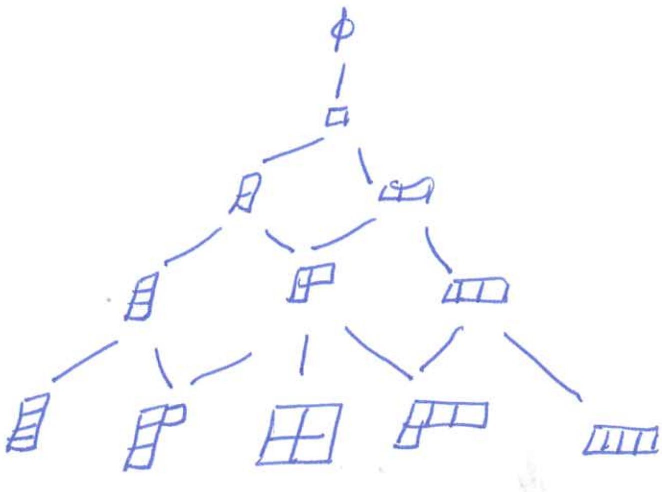
A partition is a collection of k-boxes in a corner.



$= (5,5,3,1,1)$

Let
$\hat{H}_k = \{\text{partitions with } k \text{ boxes}\}$.

The Bratelli diagram for the tower of algebras
$H_1 \subseteq H_2 \subseteq H_3 \subseteq \ldots$
has
vertices on level k: $\hat{H}_k$
edges $\lambda - \mu$ if μ is obtained from λ by adding a box.



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Example $H_5^{(2,2,1)}$ has basis

$$v_{\begin{smallmatrix}1&2\\3&4\\5\end{smallmatrix}},\ v_{\begin{smallmatrix}1&2\\3&5\\4\end{smallmatrix}},\ v_{\begin{smallmatrix}1&3\\2&4\\5\end{smallmatrix}},\ v_{\begin{smallmatrix}1&3\\2&5\\4\end{smallmatrix}},\ v_{\begin{smallmatrix}1&4\\2&5\\3\end{smallmatrix}} \quad \text{and}$$

$$T_2 v_{\begin{smallmatrix}1&2\\3&5\\4\end{smallmatrix}} = \frac{q-q^{-1}}{1-q^{2(1-(-1))}} v_{\begin{smallmatrix}1&2\\3&5\\4\end{smallmatrix}} + \left(q^{-1}+\frac{q-q^{-1}}{1-q^{2(1-(-1))}}\right) v_{\begin{smallmatrix}1&3\\2&5\\4\end{smallmatrix}}$$

$$T_2 v_{\begin{smallmatrix}1&4\\2&5\\3\end{smallmatrix}} = \frac{q-q^{-1}}{1-q^{2(-1-(-2))}} v_{\begin{smallmatrix}1&4\\2&5\\3\end{smallmatrix}} + \left(q^{-1}+\frac{q-q^{-1}}{1-q^{2}}\right) v_{\begin{smallmatrix}1&4\\3&5\\2\end{smallmatrix}}$$

$$= -q^{-1} v_{\begin{smallmatrix}1&4\\2&5\\3\end{smallmatrix}}.$$

Proof Step 1: H_k^λ is an irreducible H_k-module.

Step 2: If M is an irreducible H_k-module then $\exists \lambda$ s.t. $M \cong H_k^\lambda$.

Step 3 $H_k^\lambda \not\cong H_k^\mu$.

Step 1 (a) H_k^λ is an H_k-module

(b) H_k^λ is irreducible.

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<u>Remark:</u> <u>Frobenius</u> <u>reciprocity</u>

Since

$$\mathrm{Hom}_{H_k}\left(\mathrm{Ind}_{H_{k-1}}^{H_k}(H_{k-1}^{\mu}), H_k^{\lambda}\right) = \mathrm{Hom}_{H_{k-1}}\left(H_{k-1}^{\mu}, \mathrm{Res}_{H_{k-1}}^{H_k}(H_k^{\lambda})\right)$$

and

$$\mathrm{Hom}_{H_k}(H_k^{\lambda}, H_k^{\nu}) = \begin{cases} 0, & \text{if } \lambda \neq \nu, \\ \mathbb{C}\cdot \mathrm{Id}, & \text{if } \lambda = \nu, \end{cases}$$

we get

$$\mathrm{Ind}_{H_{k-1}}^{H_k}(H_{k-1}^{\mu}) = \bigoplus_{\substack{\lambda \supseteq \mu \\ \lambda/\mu = \square}} H_k^{\lambda}$$

<u>Note:</u> $\dim(H_k^{\lambda}) = \#$ of paths $\emptyset \to \cdots \to \lambda$.

<u>Example</u> As vector spaces

$$H_4^{(3,1)} = H_3^{(2,1)} \oplus H_3^{(3)} = H_2^{(1,1)} \oplus H_2^{(2)} \oplus H_2^{(2)}$$

$$= H_1^{(1)} \oplus H_1^{(1)} \oplus H_1^{(1)}$$ and ($H_1 \cong M_1(\mathbb{C})$ which has a one 1-dim'l simple module)

A <u>standard</u> <u>tableau</u> of shape λ is a filling T of the boxes of λ with $1, 2, \ldots, k$ such that

(a) the rows increase left to right,
(b) the columns increase top to bottom.

A <u>standard</u> <u>tableau</u> <u>of shape λ</u> is a filling T of the boxes of λ with $1, 2, \ldots, k$ such that

(a) the rows increase left to right,
(b) the columns increase top to bottom.

There is a bijection

$$\left\{\begin{matrix}\text{standard tableaux}\\ \text{of shape } \lambda\end{matrix}\right\} \xleftrightarrow{1\text{-}1} \left\{\begin{matrix}\text{paths } \phi \to \cdots \to \lambda\\ \text{in } \hat{H}\end{matrix}\right\}$$

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} \qquad \longleftarrow\!\!\!\mid \qquad \phi \to$$ 

So
$$\dim(H_k^\lambda) = \mathrm{Card}\left(\left\{\begin{matrix}\text{standard tableaux}\\ \text{of shape } \lambda\end{matrix}\right\}\right).$$

<u>Theorem</u> The irreducible H_k-modules are

$$H_k^\lambda = \mathrm{span}\{v_T \mid T \text{ is a standard tableau of shape } \lambda\}$$

with H_k-action given by

$$T_i v_T = \frac{q-q^{-1}}{1-q^{2(c(T(i))-c(T(i+1)))}} v_T + \left(q^{-1} + \frac{q-q^{-1}}{1-q^{2(c(T(i))-c(T(i+1)))}}\right) v_{s_iT}$$

where

$T(i)$ is the box containing i in T,
$c(b) = s-r$ if b is in row r and column s,
s_iT is T except with i and $i+1$ switched,
$v_{s_iT} = 0$ if s_iT is not standard.

<u>Example</u> $H_5^{\#}$ has basis

$$v_{\substack{12\\34\\5}} \quad v_{\substack{13\\24\\5}} \quad v_{\substack{12\\35\\4}} \quad v_{\substack{13\\25\\4}} \quad v_{\substack{14\\25\\3}} \quad \text{and}$$

$$T_2 v_{\substack{12\\35\\4}} = \frac{q-q^{-1}}{1-q^{2(1-(-1))}} v_{\substack{12\\35\\4}} + \left(q^{-1} + \frac{q-q^{-1}}{1-q^{2(1-(-1))}}\right) v_{\substack{13\\25\\4}}$$

$$T_2 v_{\substack{14\\25\\3}} = \frac{q-q^{-1}}{1-q^{2(-1-(-2))}} v_{\substack{14\\25\\3}} + \left(q^{-1} + \frac{q-q^{-1}}{1-q^{2(-1-(-2))}}\right) v_{\substack{14\\35\\2}}$$

$$= -q^{-1} v_{\substack{14\\25\\3}}$$

<u>Claim</u>

$$y^{\varepsilon_i^\vee} v_T = q^{c(T(i))} v_T.$$

<u>Proof</u> To show: $y^{\varepsilon_k^\vee} v_T = q^{c(T(k))} v_T$

Proof by induction:

<u>Base case $k=1$:</u> $y^{\varepsilon_1^\vee} = 1$ and $y^{\varepsilon_1^\vee} v_1 = v_1 = q^{c(T(1))} v_1 = q^0 v_1$

<u>Induction step:</u>

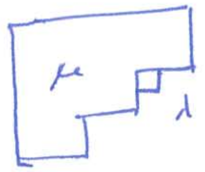
$$y^{\varepsilon_k^\vee} v_T = T_{k-1} y^{\varepsilon_{k-1}^\vee} T_{k-1} v_T$$

$$= T_{k-1} y^{\varepsilon_{k-1}^\vee} \left(\frac{q-q^{-1}}{1-q^{2(c(T(k-1))-c(T(k)))}} v_T + \left(q^{-1} + \frac{q-q^{-1}}{1-q^{2(c(T(k-1))-c(T(k)))}}\right) v_{s_{k-1}T} \right)$$

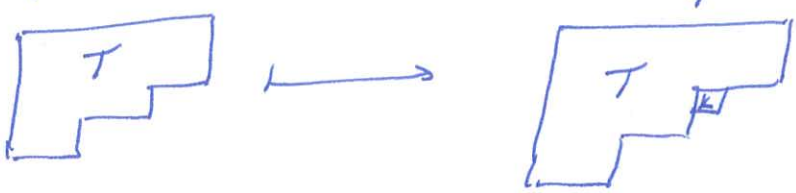
$$= T_{k-1} \left(\frac{q^{c(T(k-1))}(q-q^{-1})}{1-q^{2(c(T(k))-c(T(k)))}} v_T + q^{c(s_{k-1}T(k-1))} \left(q^{-1} + \frac{q-q^{-1}}{\sim}\right) v_{s_{k-1}T} \right)$$

<u>Claim</u>

$$\mathrm{Res}^{H_k}_{H_{k-1}}(H_k^\lambda) \cong \bigoplus_{\substack{\mu \subseteq \lambda \\ \lambda/\mu = \square}} H_{k-1}^\mu .$$

since, if  then the map

$$\left\{\begin{matrix}\text{standard tableaux} \\ \text{of shape } \mu\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{standard tableaux of} \\ \text{shape } \lambda \text{ with } k \text{ in} \\ \text{the box } \lambda/\mu\end{matrix}\right\}$$



is a bijection.

Proof of the main theorem

<u>Step 1</u> H_k^λ is an irreducible H_k-module.

<u>Step 1a</u> H_k^λ is an H_k-module.

<u>Step ~~1~~b</u> H_k^λ is simple

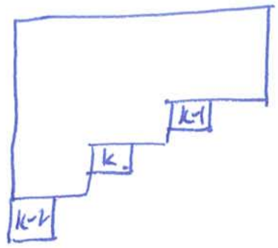
<u>Step 2</u> If $\lambda \neq \mu$ then $H_k^\lambda \not\cong H_k^\mu$

<u>Step 3</u> If M is a simple H_k-module then there exists $\lambda \in \hat{H}_k$ such that $M \cong H_k^\lambda$.

<u>Step 1a</u> To show: (aa) $T_{k-2}T_{k-1}T_{k-2}v_T = T_{k-1}T_{k-2}T_{k-1}v_T$.

(ab) $T_{k-1}^2 v_T = (q-q^{-1})T_{k-1}v_T + v_T$.

(aa) Case (aaa):



Then

$T_{k-2}T_{k-1}T_{k-2}v_T$ is a linear combination of

$v_T,\ v_{s_{k-1}T},\ v_{s_{k-2}T},\ v_{s_{k-1}s_{k-2}T},\ v_{s_{k-2}s_{k-1}T},\ v_{s_{k-1}s_{k-2}s_{k-1}T}$.

and on the span of these 6 basis elements

$$T_{k-2} = \begin{pmatrix} \frac{q-q^{-1}}{1-q^{2(a-b)}} & 0 & q^{-1}+\frac{q-q^{-1}}{1-q^{2(b-a)}} & 0 & 0 & 0 \\ 0 & \frac{q-q^{-1}}{1-q^{2(a-c)}} & 0 & 0 & q^{-1}+\frac{q-q^{-1}}{1-q^{2(c-a)}} & 0 \\ q^{-1}+\frac{q-q^{-1}}{1-q^{2(a-b)}} & 0 & \frac{q-q^{-1}}{1-q^{2(b-a)}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{q-q^{-1}}{1-q^{2(b-c)}} & 0 & q^{-1}+\frac{q-q^{-1}}{1-q^{2(c-b)}} \\ 0 & q^{-1}+\frac{q-q^{-1}}{1-q^{2(a-c)}} & 0 & 0 & \frac{q-q^{-1}}{1-q^{2(c-a)}} & 0 \\ 0 & 0 & 0 & q^{-1}+\frac{q-q^{-1}}{1-q^{2(b-c)}} & 0 & \frac{q-q^{-1}}{1-q^{2(c-b)}} \end{pmatrix}$$

where $a = c(T(k-2))$, $b = c(T(k-1))$, $c = c(T(k))$.

Then write down the matrix of T_{k-1} and

check that $T_{k-1}T_{k-2}T_{k-1} = T_{k-2}T_{k-1}T_{k-2}$,

by multiplying the matrices.

Step 1b H_k^λ is simple.

To show: If N is a submodule of H_k^λ and $N \neq 0$ then
$$N = H_k^\lambda .$$

Let $n \in N$ with $n \neq 0$. Then $n = \sum_{S \in \hat{H}_k^\lambda} c_S v_S$.
Let $T \in \hat{H}_k^\lambda$ be such that $c_T \neq 0$.
Let
$$p_T = \prod_{\substack{S \in \hat{H}_k^\lambda \\ S \neq T}} \frac{Y^{\varepsilon_i} - q^{c(S(i))}}{q^{c(T(i))} - q^{c(S(i))}}$$

Then
$$p_T v_T = v_T \quad \text{and} \quad p_T v_S = 0 \text{ if } S \neq T.$$

So $p_T n = c_T v_T$. So $v_T \in N$.

If $s_i T$ is standard then $q^{-1} + \dfrac{q - q^{-1}}{1 - q^{2(c(T(i)) - c(T(i+1)))}} \neq 0$

and
$$\left(T_i - \frac{q - q^{-1}}{1 - q^{2(c(T(i)) - c(T(i+1)))}} \right) v_T = \left(q^{-1} + \frac{q - q^{-1}}{1 - q^{2(c(T(i)) - c(T(i+1)))}} \right) v_{s_i T}$$

So $v_{s_i T} \in n$.

If $i+1$ is northeast of i in T then apply s_i to T.
This process will reduce T to

$$R = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 & 9 & 10 \\ \hline 11 & 12 & 13 \\ \cline{1-3} 14 \\ \cline{1-1} \end{array}$$ the reading tableau.

So $v_R = v_{s_{i_\ell} s_{i_{\ell-1}} \cdots s_{i_1} T} \in N$.

<u>Step 3</u>

<u>To show</u>:
$$H_k \simeq \bigoplus_{\lambda \in \hat{H}_k} M_{d_\lambda}(\mathbb{C})$$

where d_λ = # of standard tableaux of shape λ.