Representation Theory 28.04.2009

Our language

$\mathfrak{h}_{\mathbb{Z}}^*$ is a $\mathbb{Z}$-vector space

$W_0 \subseteq GL(\mathfrak{h}_{\mathbb{Z}}^*)$ a finite group generated by reflections s_α, $\alpha \in R^+$, with

$$s_\alpha \mu = \mu - \langle \mu, \alpha^\vee \rangle \alpha, \quad \text{for } \mu \in \mathfrak{h}_{\mathbb{Z}}^*$$

C is a fundamental chamber for the W_0-action on $\mathfrak{h}_{\mathbb{R}}^* = \mathbb{R} \otimes_{\mathbb{Z}} \mathfrak{h}_{\mathbb{Z}}^*$;

$$W_0 \overset{1-1}{\longleftrightarrow} \{\text{chambers in } \mathfrak{h}_{\mathbb{R}}^*\}$$

$\mathfrak{h}^{\alpha_1^\vee}, \ldots, \mathfrak{h}^{\alpha_n^\vee}$ are the walls of C and their reflections $s_1, \ldots, s_n$ are the simple reflections

$\mathbb{C}[X] = \text{span}\{X^\mu \mid \mu \in \mathfrak{h}_{\mathbb{Z}}^*\}$ with

$$X^\mu X^\nu = X^{\mu+\nu} \quad \text{and} \quad w X^\mu = X^{w\mu}$$

$\mathbb{C}[X]^{W_0} = \{ f \in \mathbb{C}[X] \mid wf = f, \text{ for all } w \in W_0 \}$

$\mathbb{C}[X]^{\det} = \{ f \in \mathbb{C}[X] \mid wf = \det(w) f, \text{ for all } w \in W_0 \}$

For $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$ define

$$m_\lambda = \sum_{\gamma \in W_0 \lambda} X^\gamma \quad \text{and} \quad a_\lambda = \sum_{w \in W_0} \det(w^{-1}) X^{w\lambda}.$$

Weyl characters

Let

$$P^{+} = \mathfrak{h}_{\mathbb{Z}}^{*} \cap \overline{C} \qquad \text{so that} \qquad (\mathfrak{h}_{\mathbb{Z}}^{*})^{+} \xrightarrow{\sim} (\mathfrak{h}_{\mathbb{Z}}^{*})^{++}$$

$$P^{++} = \mathfrak{h}_{\mathbb{Z}}^{*} \cap C \qquad\qquad \lambda \longmapsto \rho + \lambda$$

as semigroups. Then

$\{m_\lambda \mid \lambda \in (\mathfrak{h}_{\mathbb{Z}}^{*})^{+}\}$ is a basis of $\mathbb{C}[X]^{W_0}$

$\{a_{\lambda+\rho} \mid \lambda+\rho \in (\mathfrak{h}_{\mathbb{Z}}^{*})^{++}\}$ is a basis of $\mathbb{C}[X]^{\det}$

and

$$\begin{matrix} \mathbb{C}[X]^{W_0} & \xrightarrow{\sim} & \mathbb{C}[X]^{\det} \\ f & \longmapsto & a_\rho f \end{matrix} \qquad \text{as } \mathbb{C}[X]^{W_0}\text{-modules}$$

The Weyl character, or Schur function, is

$$s_\lambda = \frac{a_{\lambda+\rho}}{a_\rho}, \qquad \text{for } \lambda \in (\mathfrak{h}_{\mathbb{Z}}^{*})^{+}$$

The Weyl denominator, or Vandermonde, is

$$a_\rho = X^{\rho} \prod_{\alpha \in R^{+}} (1 - X^{-\alpha}).$$

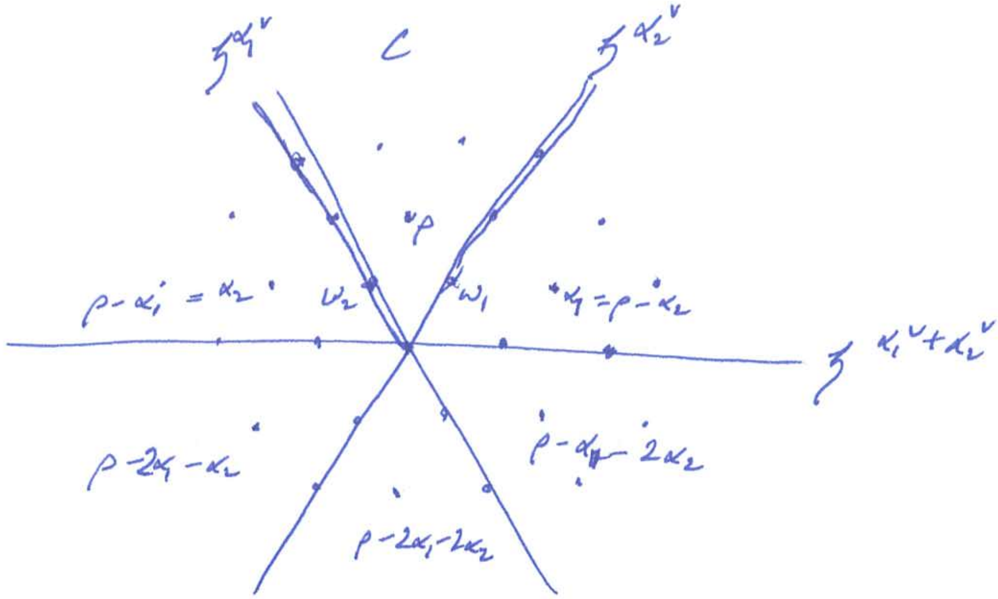
The Kostka numbers or weight multiplicities are

$K_{\lambda\mu}$ given by $\displaystyle s_\lambda = \sum_{\mu \in (\mathfrak{h}_{\mathbb{Z}}^{*})^{+}} K_{\lambda\mu} m_\mu$

for $\lambda, \mu \in (\mathfrak{h}_{\mathbb{Z}}^{*})^{+}$

Example sl_3 $\quad \mathfrak{h}^*_{\mathbb{Z}} = \mathrm{span}\{\omega_1, \omega_2\}$

$$W_0 = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1,\ s_1s_2s_1 = s_2s_1s_2 \rangle$$



$$a_\rho = X^\rho (1 - X^{-\alpha_1})(1 - X^{-\alpha_2})(1 - X^{-(\alpha_1+\alpha_2)})$$

$$= X^\rho - X^{\rho-\alpha_1} - X^{\rho-\alpha_2} - X^{\rho-(\alpha_1+\alpha_2)} + X^{\rho-(\alpha_1+\alpha_2)} +$$

$$+ X^{\rho-2\alpha_1-\alpha_2} + X^{\rho-\alpha_1-2\alpha_2} - X^{\rho-2\alpha_1-2\alpha_2}$$

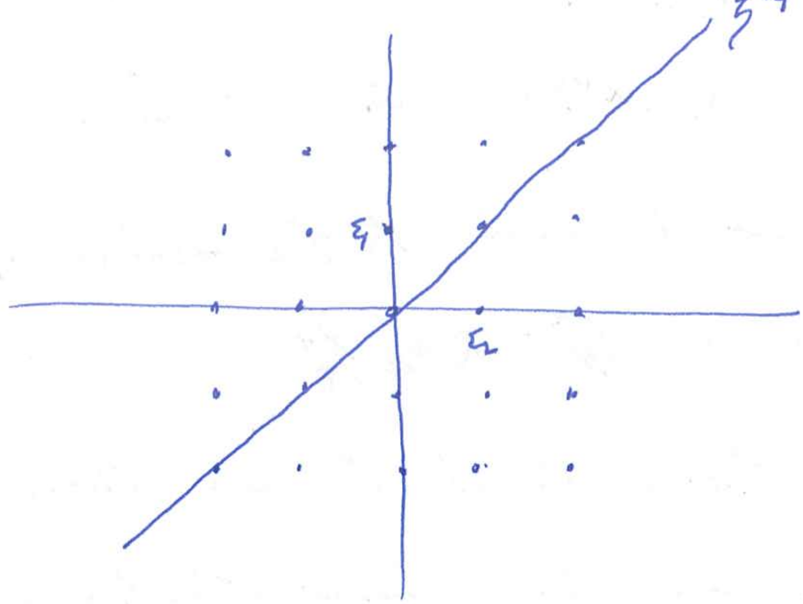
$$s_\rho = \frac{a_{\rho+\rho}}{a_\rho} = \frac{X^{2\rho} - X^{s_1 2\rho} - X^{s_2 2\rho} - X^{s_1s_2s_1 2\rho} + X^{s_1s_2 2\rho} + X^{s_2s_1 2\rho}}{X^\rho - X^{s_1\rho} - X^{s_2\rho} - X^{s_1s_2s_1\rho} + X^{s_1s_2\rho} + X^{s_2s_1\rho}}$$

$$= X^\rho + X^{s_1\rho} + X^{s_2\rho} + X^{s_1s_2\rho} + X^{s_2s_1\rho} + X^{s_1s_2s_1\rho} + 2.$$

Example GL_n

$$\mathfrak{h}^*_{\mathbb{Z}} = \sum_{i=1}^{n} \mathbb{Z}\varepsilon_i \quad \text{and} \quad W_0 = S_n$$

acting by permuting $\varepsilon_1, \ldots, \varepsilon_n$



$W_0 = S_2 = \langle s_1 \mid s_1^2 = 1 \rangle$

$$C = \{ \lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \in \mathfrak{h}^*_{\mathbb{R}} \mid \lambda_1 > \lambda_2 > \cdots > \lambda_n \}$$

$$\bar{C} = \{ \lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \in \mathfrak{h}^*_{\mathbb{R}} \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \}.$$

$$\rho = (n-1)\varepsilon_1 + (n-2)\varepsilon_2 + \cdots + \varepsilon_{n-1}$$

$$\mathbb{C}[X] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \quad \text{where} \quad x_i = X^{\varepsilon_i}.$$

$$a_\rho = \sum_{w \in S_n} \det(w^{-1})\, w(x_1^{n-1} x_2^{n-2} \cdots x_{n-1})$$

$$= \det \begin{pmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & x_2 & 1 \\ \vdots & & \ddots & & \vdots \\ x_n^{n-1} & x_n^{n-1} & \cdots & x_n & 1 \end{pmatrix} = \prod_{1 \leq i < j \leq n} (x_i - x_j)$$

If $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)^+$ then $\lambda = \lambda_1 \varepsilon_1 + \cdots + \lambda_n \varepsilon_n$ with $\lambda_1 \geq \cdots \geq \lambda_n$ and

$$s_\lambda = \frac{a_{\lambda+\rho}}{a_\rho} = \frac{\sum_{w \in S_n} \det(w^{-1})\, w\left(x_1^{\lambda_1+n-1} x_2^{\lambda_2+n-2} \cdots x_{n-1}^{\lambda_{n-1}+1} x_n^{\lambda_n}\right)}{\sum_{w \in S_n} \det(w^{-1})\, w\left(x_1^{n-1} x_2^{n-2} \cdots x_{n-1}\right)}$$

$$= \frac{\det\begin{pmatrix} x_1^{\lambda_1+n-1} & x_1^{\lambda_2+n-2} & \cdots & x_1^{\lambda_{n-1}+1} & x_1^{\lambda_n} \\ x_2^{\lambda_1+n-1} & x_2^{\lambda_2+n-2} & \cdots & x_2^{\lambda_{n-1}+1} & x_2^{\lambda_n} \\ \vdots & \vdots & & \vdots & \vdots \\ x_n^{\lambda_1+n-1} & x_n^{\lambda_2+n-2} & \cdots & x_n^{\lambda_{n-1}+1} & x_n^{\lambda_n} \end{pmatrix}}{\det\begin{pmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & x_2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ x_n^{n-1} & x_n^{n-2} & \cdots & x_n & 1 \end{pmatrix}}$$