Representation Theory 27.05.2009

Let G be a group and B a subgroup.

Then

$$\mathbb{1}_B^G = \mathrm{Ind}_B^G(\mathrm{triv}) = \mathrm{span}\{ g \otimes \mathbb{1} \mid g \in G \}$$ with

$$gb \otimes \mathbb{1} = g \otimes \mathbb{1}, \quad \text{and}$$

G action given by

$$g(h \otimes \mathbb{1}) = gh \otimes \mathbb{1}, \quad \text{so that}$$

$$\mathbb{1}_B^G = \mathrm{span}\{ x \otimes \mathbb{1} \mid x \in \widehat{G/B} \}$$ where

$\widehat{G/B}$ is a set of representatives of the cosets ~~in~~ G/B

$$G = \bigsqcup_{x \in \widehat{G/B}} xB$$

The Hecke algebra of $B \subseteq G$ is

$$Z = \mathrm{End}_G(\mathbb{1}_B^G)$$

Example When $B = \{1\}$

$$\begin{aligned} \mathbb{1}_B^G &\xrightarrow{\sim} \mathbb{C}G \\ g \otimes 1 &\longmapsto g \end{aligned} \qquad \text{and} \qquad \begin{aligned} Z &\xrightarrow{\sim} \mathbb{C}G \\ R_g &\longleftarrow g \end{aligned}$$

Let

$$v_1 = \frac{1}{|B|}\sum_{b\in B} b \quad \text{(an element of } \mathbb{C}G\text{)}.$$

Then

$$v_x = x v_1 = \frac{1}{|B|}\sum_{b\in B} xb, \quad \text{for } x\in G$$

and note that $v_{xb} = v_x$.

Then

$$\begin{aligned} \mathcal{H}_B^G &\xrightarrow{\sim} (\mathbb{C}G)v_1 \\ x\otimes \mathbb{1} &\longmapsto x v_1 \end{aligned}$$

Proposition
$$\begin{aligned} Z &\xrightarrow{\sim} v_1(\mathbb{C}G)v_1 \\ R_{v_1wv_1} &\longleftarrow\!\shortmid v_1wv_1 \end{aligned}$$

Proof (a) $R_{v_1wv_1}\in Z$.

If $hv_1\in(\mathbb{C}G)v_1$, $g\in G$ then

$$\begin{aligned} R_{v_1wv_1}g\cdot hv_1 &= ghv_1v_1wv_1 = \cancel{ghv_1wv_1} \\ &= gR_{v_1wv_1}\cdot hv_1 \end{aligned}$$

(b) Assume $\varphi\in Z$, $\varphi: \mathbb{C}Gv_1 \to \mathbb{C}Gv_1$
$$v_1 \longmapsto \varphi(v_1)$$

~~and~~ ~~$\varphi(v_1)$~~ Let $hv_1\in\mathbb{C}Gv_1$. Then

$$\varphi(hv_1) = h\varphi(v_1) = hv_1\varphi(v_1) = R_{\varphi(v_1)}hv_1 \;//$$

Let W_0 be a set of coset representatives of the B-double cosets in G.

$$G = \bigsqcup_{w \in W_0} BwB.$$

Then

$$Z = \mathrm{span}\{\tilde{T}_w \mid w \in W_0\} \quad \text{where } \tilde{T}_w = v_1 w v_1$$

<u>Example</u> $SL_2(\mathbb{F}_q) = G$ and $B = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\}$.

$x_\alpha(c) = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$, $x_{-\alpha}(z) = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$, $n_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$h_\alpha(d) = \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix}$.

Then

$$G = B \sqcup \left(\bigsqcup_{c \in \mathbb{F}_q} x_\alpha(c) n_1 B \right)$$

$$= n_1 B \sqcup \left(\bigsqcup_{z \in \mathbb{F}_q} \underline{x}_\alpha(z) n_1 B \right)$$

with

$$x_\alpha(c) n_1 = x_{-\alpha}(c^{-1}) x_\alpha(c) h_\alpha(-c)$$

since $\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -c & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ c^{-1} & 1 \end{pmatrix}\begin{pmatrix} c & 1 \\ 0 & c^{-1} \end{pmatrix} = \Big(c$

$$\begin{pmatrix} 1 & 0 \\ c^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -c & 0 \\ 0 & -c^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c^{-1} & 1 \end{pmatrix}\begin{pmatrix} -c & 1 \\ 0 & -c^{-1} \end{pmatrix} = \begin{pmatrix} -c & 1 \\ -1 & 0 \end{pmatrix}$$

Then $W_0 = \{1, n_1\}$ with

$$G = B \sqcup B n_1 B$$

Let $\hat{T}_1 = v_1 = \frac{1}{|B|} \sum_{b \in B} b$ and

$$\hat{T}_{s_1} = v_1 n_1 v_1 = \sum_{c \in \mathbb{F}_q} x_\alpha(c) n_1 v_1 = \frac{1}{|B|} \sum_{y \in B n_1 B} y.$$

Then

$$x_\alpha(z) n_1 v_1 \cdot \hat{T}_{s_1} = x_\alpha(z) n_1 v_1 n_1 v_1$$

$$= x_\alpha(z) n_1 \sum_{c \in \mathbb{F}_q} x_\alpha(c) n_1 v_1$$

$$= x_\alpha(z) n_1 n_1 v_1 + \sum_{c \in \mathbb{F}_q^\times} x_\alpha(z) n_1 x_\alpha(c) n_1 v_1$$

Then $n_1 x_\alpha(c) n_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$= \begin{pmatrix} -1 & 0 \\ c & -1 \end{pmatrix} = x_{-\alpha}(-c) \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

So

$$x_\alpha(z) n_1 v_1 = x_\alpha(z) v_1 + \sum_{c \in \mathbb{F}_q^\times} x_\alpha(z) x_{-\alpha}(-c) v_1$$

$$= v_1 + \sum_{c \in \mathbb{F}_q^\times} x_\alpha(z) x_\alpha(+c^{-1}) n_1 v_1$$

$$= v_1 + \sum_{c \in \mathbb{F}_q^\times} x_\alpha(z + c^{-1}) n_1 v_1 = v_1 + \sum_{z_1 \neq z} x_\alpha(z_1) n_1 v_1$$