Representation Theory 26.05.2009

Review of restriction and induction

Let $C \subseteq A$, so that C is a subalgebra of A.

Let M be an A-module. Then $\mathrm{Res}_C^A(M)$ is

M with C acting as a subset of A.

Let N be a C-module. Then

$$\mathrm{Ind}_C^A(N) = A \otimes_C N = \mathrm{span}\{a \otimes n \mid a \in A, n \in N\}$$

with the relations

$$(a_1 + a_2) \otimes n = a_1 \otimes n + a_2 \otimes n,$$

$$a \otimes (n_1 + n_2) = a \otimes n_1 + a \otimes n_2,$$

$$ac \otimes n = a \otimes cn,$$

for $a, a_1, a_2 \in A$, $n, n_1, n_2 \in N$ and $c \in C$.

Let B be a subgroup of G. Let

$\mathrm{triv} = \mathrm{span}\{\mathbb{1}\}$ with $b\mathbb{1} = \mathbb{1}$, for $b \in B$,

so that triv is a 1-dimensional B-module.

Then

$$\mathrm{Ind}_B^G(\mathrm{triv}) = \mathbb{C}G \otimes_{\mathbb{C}B} \mathbb{1}$$

$$= \mathrm{span}\{g \otimes \mathbb{1} \mid g \in G\},$$ with the

relation $gb \otimes \mathbb{1} = g \otimes b\mathbb{1}$, for $g \in G$, $b \in B$.

Let $\widehat{G/B}$ be a set of coset representatives of the cosets in G/B so that

$$G = \bigsqcup_{x \in \widehat{G/B}} xB$$

Then $\mathrm{Ind}_B^G(\mathrm{triv}) = \mathrm{span}\{ x \otimes \mathbb{1} \mid x \in \widehat{G/B}\}$.

Let

$v_x = x\mathbb{1} = x \otimes \mathbb{1}$, to increase our notational confusion.

The <u>Hecke algebra</u> of $B \subseteq G$ is

$$Z = \mathrm{End}_G(\mathrm{Ind}_B^G(\mathrm{triv})) = \mathrm{End}_G(\mathbb{1}_B^G)$$

Recall that, as a $(\mathbb{C}G, Z)$ bimodule

$$\mathbb{1}_B^G = \bigoplus_{\lambda \in \hat{Z}} G^\lambda \otimes Z^\lambda.$$

where G^λ are simple G-modules,

Z^λ are simple Z-modules

<u>Example</u> If $B = \{1\}$ then $\mathbb{1}_B^G = \mathrm{span}\{ g \otimes \mathbb{1} \mid g \in G\}$

and
$$\begin{array}{ccc} \mathbb{1}_B^G & \xrightarrow{\sim} & \mathbb{C}G \\ g \otimes \mathbb{1} & \longmapsto & g \end{array}$$
with $g \cdot h = gh$, for $g \in G$, $h \in \mathbb{C}G$.

<u>Proposition</u> Then
$$\begin{array}{ccc} Z & \xrightarrow{\sim} & \mathbb{C}G \\ R_g & \longleftarrow & g \end{array}$$
where
$$\begin{array}{cccc} R_g : & \mathbb{C}G & \to & \mathbb{C}G \\ & h & \longmapsto & hg. \end{array}$$

Proof (a) $R_g \in Z$.

If $h \in G$, ~~then~~ and $k \in G$ then

$$R_g k(h) = R_g(kh) = khg = k \cdot hg = k R_g(h).$$

(b) Assume $\varphi : \mathbb{C}G \to \mathbb{C}G$ and $\varphi \in Z$.

Then $\varphi(1) \in \mathbb{C}G$ and

$$\varphi(g) = \varphi(g \cdot 1) = g\varphi(1) = R_{\varphi(1)}(g).$$

So $\varphi = R_{\varphi(1)}$. Thus $R : \mathbb{C}G \longrightarrow Z$ is surjective. //

In general, let

$$v_x = \sum_{y \in xB} y \ \in \mathbb{C}G, \text{ so that } v_1 = \sum_{b \in B} b$$

and $bv_1 = v_1$, for $b \in B$. Then $v_x = x v_1$

$$\begin{aligned} \mathbb{C}Gv_1 = \operatorname{span}\{v_x \mid x \in G/B\} &\longrightarrow 1_B^G \\ v_x &\longmapsto x \otimes 1 \end{aligned}$$

is a G-module isomorphism.

So $$1_B^G = \mathbb{C}Gv_1.$$

Proposition Let W be a set of coset representatives of the B-double cosets in G so that

$$G = \bigsqcup_{w \in W} BwB. \quad \text{Let } T_w = \sum_{y \in BwB} y.$$

Then, if $Z = End_G(\mathbb{1}_B^G)$ then (4)

$$v_1 \mathbb{C}G v_1 = \operatorname{span}\{T_w \mid w \in W\} \xrightarrow{\sim} Z$$
$$T_w \longmapsto R_{T_w}$$

<u>Proof</u> (a) $R_{T_w} \in Z$.

Let $g \in G$, $x \in \hat{G/B}$, $w \in W$. Then

$$R_{T_w} g(v_x) = R_{T_w}(g v_x) = g v_x T_w = g \cdot R_{T_w}(v_x).$$

(b) Assume $\varphi: \mathbb{1}_B^G \to \mathbb{1}_B^G$ is in Z.

Then $\varphi(v_1) \in \mathbb{1}_B^G$ and

$$\varphi(v_x) = \varphi(x v_1) = x\varphi(v_1) = x v_1 \varphi(v_1) \frac{1}{|B|}$$
$$= v_x \frac{1}{|B|} \varphi(v_1) = R_{\frac{1}{|B|}\varphi(v_1)}(v_x).$$

So $\varphi = R_{\frac{1}{|B|}\varphi(v_1)}$. Note that

$$\frac{1}{|B|}\varphi(v_1) \cancel{\in \mathbb{C}G v_1 \text{ and}} = v_1 \frac{1}{|B|^2}\varphi(v_1) \in v_1 (\mathbb{C}G) v_1. //$$

<u>Example</u> $G = GL_n(\mathbb{F}_q)$ and $B = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \in GL_n(\mathbb{F}_q) \right\}$

Then ~~G/B~~ cosets ~~representat~~ in G/B are called flags, and representatives are the elements of

$$\hat{G/B} = \{\text{invertible echelon matrices}\}.$$

$$GL_2(\mathbb{F}_2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, ad - bc \neq 0 \right\}$$

Let $x_\alpha(c) = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$, $x_{-\alpha}(c) = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$, $s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Then

$$x_\alpha(c) s_1 = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ c^{-1} & 1 \end{pmatrix} \begin{pmatrix} c & 1 \\ 0 & -c^{-1} \end{pmatrix} = x_{-\alpha}(-c^{-1}) h_1(c) h_2(-c^{-1}).$$

So

$$G/B = \{B\} \cup \{ x_\alpha(c) s_1 B \mid c \in \mathbb{F}_q \}$$

$$= \{ x_{-\alpha}(z) B \mid z \in \mathbb{F}_q \} \cup \{ s_1 B \}.$$

Then

$$G = \bigsqcup_{w \in W_0} BwB \quad \text{with} \quad W_0 = \{1, s_1\}$$

and

$$\hat{T}_{s_1} \cdot v_{x_\alpha(c) s_1} = x_\alpha(c) s_1 v_1 \cdot s_1 v_1$$

$$= x_\alpha(c) s_1 x_\alpha(z) s_1 B$$

$$= x_\alpha(c) s_1 x_{-\alpha}(-c^{-1})$$

Let $G = GL_n(\mathbb{F}_q)$, where

$\mathbb{F}_q$ is the finite field with q elements.

Let

$$B = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \in GL_n \right\} \quad \text{and} \quad T = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \in GL_n \right\}.$$

Let $\mathbb{C}_1 = \mathrm{span}\{v_1\}$ be the trivial B-module, so that

$$bv_1 = v_1, \quad \text{for } b \in B.$$

Then

$$\mathrm{Ind}_B^G(\mathbb{C}_1) = \mathrm{span}\{ gv_1 \mid \cancel{g \text{ is a}}\ g \in G\}$$

with $\quad gb \otimes v_1 = g \otimes bv_1 \quad$ for $b \in B$.

~~with~~ and G-action given by

$$g(h \otimes v_1) = gh \otimes v_1, \quad \text{for } g, h \in G.$$

The Hecke algebra of $B \subseteq G$ is

$$H = \mathrm{End}_G(\mathrm{Ind}_B^G(\mathbb{C}_1)).$$

~~What does~~

Let $\widehat{G/B}$ be a set of coset representatives of the cosets in G/B so that

$$G = \bigsqcup_{x \in \widehat{G/B}} xB$$

Let

$$1_B^G = \mathrm{span}\{ v_x \mid x \in \widehat{G/B} \} \quad \text{with}$$

$$v_x = \sum_{y \in xB} y, \quad \text{for } x \in \widehat{G/B}$$

So that $1_B^G \subseteq \mathbb{C}G$. Let W_0 be a set of representative of the double cosets of B in G so that

$$G = \bigsqcup_{w \in W} BwB.$$

Let

$$T_w = \sum_{z \in BwB} z \quad \text{and} \quad H = \mathrm{span}\{T_w \mid w \in W\}.$$

so that $H \subseteq \mathbb{C}G$.

Then

$$1_B^G = \mathbb{C}G v_1 \quad \text{and} \quad H = v_1 \mathbb{C}G v_1$$

$$= \frac{(K_1 - K_1^{-1})(K_2 - K_2^{-1})}{(q-q^{-1})^3}\left(-q^{-2}(q+q^{-1})K_1K_2 + q^2(q+q^{-1})K_1^{-1}K_2^{-1}\right)$$

$$= (q+q^{-1})(K_1 - K_1^{-1})(K_2 - K_2^{-1})(-q^{-2}K_1K_2 + q^2K_1^{-1}K_2^{-1})$$

$$GL_2(\mathbb{F}_q)/B = \{B\} \cup \{x_\alpha(z)s_1B \mid z \in \mathbb{F}_q\}.$$

Then

$$\cancel{Bs_1B \cdot x_\alpha(z)s_1B = \sum_{z_1 \in \mathbb{F}_q}' x_\alpha(z_1)s_1B \cdot x_\alpha(z)s_1B}$$

$$\cancel{= \sum_{\substack{z_1 \in \mathbb{F}_q \\ z_2 \in \mathbb{F}_q}}' x_\alpha(z_1)s_1x_\alpha(z_2)s_1B}$$

$$x_\alpha(z)s_1B \cdot Bs_1B = \sum_{z_1 \in \mathbb{F}_q}' x_\alpha(z)s_1x_\alpha(z_1)s_1B$$

$$= x_\alpha(z)s_1s_1B + \sum_{z_1 \in \mathbb{F}_q^\times} x_\alpha(z)s_1x_\alpha(z_1)s_1B$$

$$= x_\alpha(z)B + \sum_{z_1 \in \mathbb{F}_q^\times} x_\alpha(z+z_1^{-1})s_1B.$$

$$= B + \sum_{\substack{z_1' \in \mathbb{F}_q \\ z_1' \neq z}}' x_\alpha(z_1')s_1B$$

~~Thus~~ So $v_z T_{s_1} = v_e + \sum_{x \neq z}' v_x$

$$\sum_{z \in \mathbb{F}_q} x_\alpha(z)s_1B \cdot Bs_1B = qB + (q-1)Bs_1B.$$

So $\tilde{T}_{s_1}\tilde{T}_{s_1} = q\tilde{T}_1 + (q-1)\tilde{T}_{s_1}$

$$x_\alpha(z)s_1 B \cdot B s_1 B = \sum_{z_1 \in \mathbb{F}_q} x_\alpha(z) s_1\, x_\alpha(z_1) s_1 B$$

$$= x_\alpha(z) s_1 s_1 B + \sum_{z_1 \in \mathbb{F}_q^\times}' x_\alpha(z) s_1\, x_\alpha(z_1) s_1 B$$

$$= x_\alpha(z) B + \sum_{z_1 \in \mathbb{F}_q^\times}' x_\alpha(z + z_1^{-1}) s_1 B$$

since

$$x_\alpha(z) s_1\, x_\alpha(z_1) s_1 = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & z_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\cancel{= \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z_1 & 1 \end{pmatrix}}$$

$$\cancel{= \begin{pmatrix} 1+zz_1 & z \\ z_1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & z+z_1^{-1} \\ 0 & 1 \end{pmatrix} \Big(1}$$

$$= x_\alpha(z)\, s_1\, x_{-\alpha}(z_1^{-1})\, h_1(z_1)\, h_2(-z_1^{-1})$$

$$= x_\alpha(z)\, x_\alpha(z_1^{-1})\, s_1\, h_1(z_1)\, h_2(-z_1^{-1})$$

$$= x_\alpha(z + z_1^{-1})\, s_1\, h_1(z_1)\, h_2(-z_1^{-1}).$$

So $$x_\alpha(z) s_1 B \cdot B s_1 B_1 = B + \sum_{\substack{z_1' \in \mathbb{F}_q \\ z_1' \neq z}} x_\alpha(z_1') s_1 B.$$

Thus

$$\hat{T}_{s_1} \cdot \hat{T}_{s_1} = q \hat{T}_1 + (q-1) T_{s_1} = \sum_{z \in \mathbb{F}_q} x_\alpha(z) s_1 B \cdot B s_1 B$$

$$= \sum_{z \in \mathbb{F}_q} \Big(B + \sum_{\substack{z_1' \in \mathbb{F}_q \\ z_1' \neq z}} x_\alpha(z_1') s_1 B \Big) = q \hat{T}_{1_B} + (q-1) \hat{T}_{s_1}.$$

$$0 \to \mathcal{O}(T^*(G/B)) \longrightarrow \mathcal{O}_{G/B} \longrightarrow \mathcal{O}_B \longrightarrow 0$$

$$\mathcal{L}_\alpha = G \times_B \mathbb{C}_{-\alpha} = \{ (g,c) \mid g, c \in \mathbb{C} \}$$

$T^*(G/B)$ has

$$\boxed{(x_\alpha(z)s_1, c) = (x_{-\alpha}(z^{-1}), z^{-2}c)}$$

and

$$\mathcal{O}_{G/B} = \mathcal{L}_0 = G \times_B \mathbb{C}_1 = \{(g,c) \mid g, c \in \mathbb{C}\}$$

has $\boxed{(x_\alpha(z)s_1, c) = (x_{-\alpha}(z^{-1}), c)}$

~~Hence~~ What's the map?