Representation Theory 25.03.2009

Adjoint functors

Let $F : \{A\text{-modules}\} \to \{B\text{-modules}\}$ be a functor.

The adjoint functor $F^\vee : \{B\text{-modules}\} \to \{A\text{-modules}\}$ is determined by

$$\mathrm{Hom}_{B\text{-mod}}(F^\vee M, N) \simeq \mathrm{Hom}_{A\text{-mod}}(M, FN)$$

The adjoint functor to Res_A^B is induction Ind_A^B

$$\mathrm{Ind}_A^B : \{A\text{-modules}\} \to \{B\text{-modules}\}$$

It is given explicitly by

$$\mathrm{Ind}_A^B(M) = B \otimes_A M$$

where $B \otimes_A M$ is generated by $b \otimes m$, $b \in B$, $m \in M$ with relations

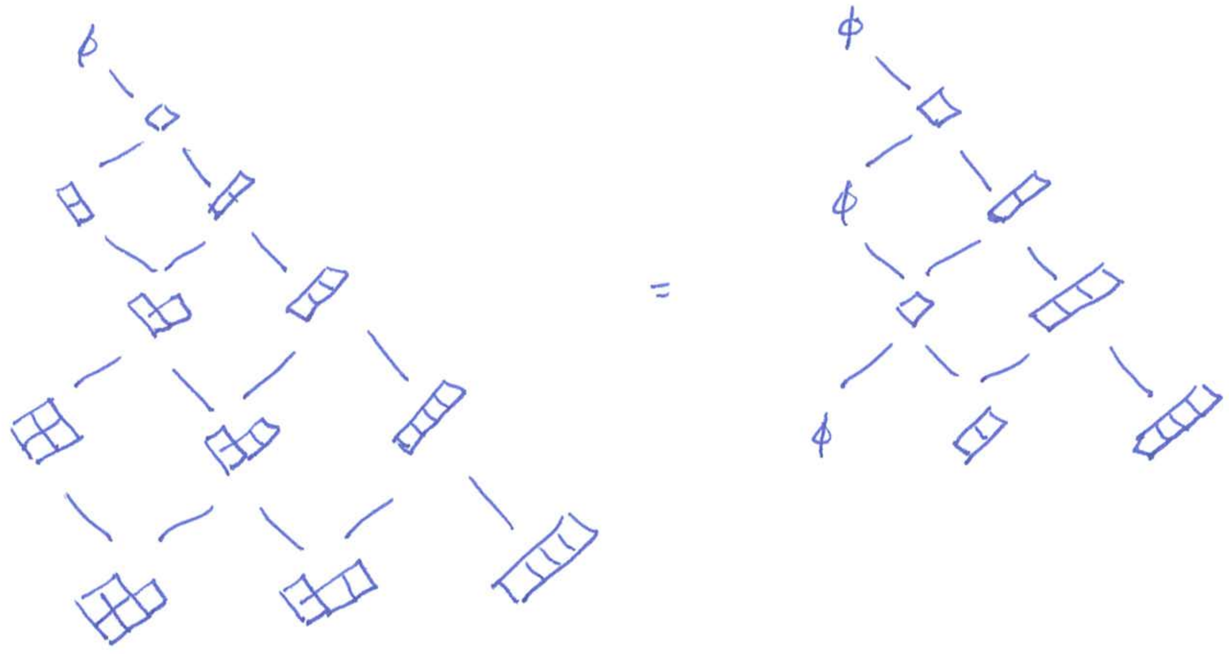
$$ba \otimes m = b \otimes am \quad \text{and bilinearity,}$$

for $b \in B$, $a \in A$, $m \in M$,

and the action of B on $B \otimes_A M$ is given by

$$b(b' \otimes m) = bb' \otimes m, \quad \text{for } b \in B,\ b' \otimes m \in B \otimes_A M.$$

The Bratelli diagram for $TL_1 \subseteq TL_2 \subseteq \cdots$ is



where the RHS has

(a) partitions with $k, k-2, k-4, \ldots$ boxes and ≤ 1 row on level k

(b) edges corresponding to adding and removing a box.

Note that

$$\mathrm{Res}^{TL_k}_{TL_{k-1}}\left(TL_k^{\lambda}\right) = \bigoplus_{\mu \in \hat{TL}_{k-1}} \left(TL_{k-1}^{\mu}\right)^{m_{\lambda\mu}}$$

implies

$$\mathrm{Ind}^{TL_k}_{TL_{k-1}}\left(TL_{k-1}^{\mu}\right) = \bigoplus_{\lambda \in \hat{TL}_k} \left(TL_k^{\lambda}\right)^{m_{\lambda\mu}}.$$

since

$$m_{\lambda\mu} = \dim\left(\mathrm{Hom}_{TL_{k-1}}\left(TL_{k-1}^{\mu}, \mathrm{Res}^{TL_k}_{TL_{k-1}}\left(TL_k^{\lambda}\right)\right)\right).$$

The algebra $U\mathfrak{sl}_2$

A <u>Lie algebra</u> is a vector space $\mathfrak{g}$ with a bracket $[\,,\,]:\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$ such that

(a) $[x,y]=-[y,x]$, for $x,y\in\mathfrak{g}$

(b) $[[x,y],z]+[[z,x],y]+[[y,z],x]=0$ for $x,y,z\in\mathfrak{g}$

A Lie algebra is not an algebra.

The <u>enveloping algebra</u> of $\mathfrak{g}$ is the algebra $U\mathfrak{g}$ generated by the vector space $\mathfrak{g}$ with

$$xy=yx+[x,y],\quad \text{for } x,y\in\mathfrak{g}$$

The <u>Lie algebra</u> $\mathfrak{sl}_2$ is the vector space

$$\mathfrak{sl}_2=\{a\in M_2(\mathbb{C})\mid \mathrm{tr}\, a=0\}$$

with bracket

$$[a,b]=ab-ba,\quad \text{for } a,b\in\mathfrak{sl}_2.$$

(where the product on the RHS is matrix mult).

<u>Proposition</u> The Lie algebra $\mathfrak{sl}_2$ is generated by

$$x=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad y=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad h=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

with relations

$$[x,y]=h,\quad [h,x]=2x,\quad [h,y]=-2y.$$

The enveloping algebra of $\mathfrak{sl}_2$ is the algebra $U\mathfrak{sl}_2$ generated by x, y, h with relations

$$xy = yx + h, \quad hx = xh + 2x, \quad hy = yh - 2y.$$

<u>HW</u>: Show that $U\mathfrak{sl}_2$ has basis

$$\{ y^{m_1} h^{m_2} x^{m_3} \mid m_1, m_2, m_3 \in \mathbb{Z}_{\geq 0} \}$$

The Lie algebra $\mathfrak{su}_2$ is

$$\mathfrak{su}_2 = \{ x \in \mathfrak{gl}_2(\mathbb{C}) \mid \mathrm{tr}\, x = 0 \text{ and } x + \bar{x}^t = 0 \}$$

$$= \mathbb{R}\text{-span}\{ i\sigma^x, i\sigma^y, i\sigma^z \}, \text{ where}$$

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$[\sigma^x, \sigma^y] = 2i\sigma^z, \quad [\sigma^y, \sigma^z] = 2i\sigma^x, \quad [\sigma^z, \sigma^x] = 2i\sigma^y.$$

Then $\mathfrak{sl}_2(\mathbb{C})$ is the <u>complexification</u> of $\mathfrak{su}_2$

$$\mathfrak{sl}_2(\mathbb{C}) = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{su}_2 = \mathbb{C}\text{-span}\{ i\sigma^x, i\sigma^y, i\sigma^z \}$$

and the change of basis is

$$\sigma^x = x + y, \quad \sigma^y = -ix + iy$$

$$x = \tfrac{1}{2}(\sigma^x + i\sigma^y), \quad y = \tfrac{1}{2}(\sigma^x - i\sigma^y).$$