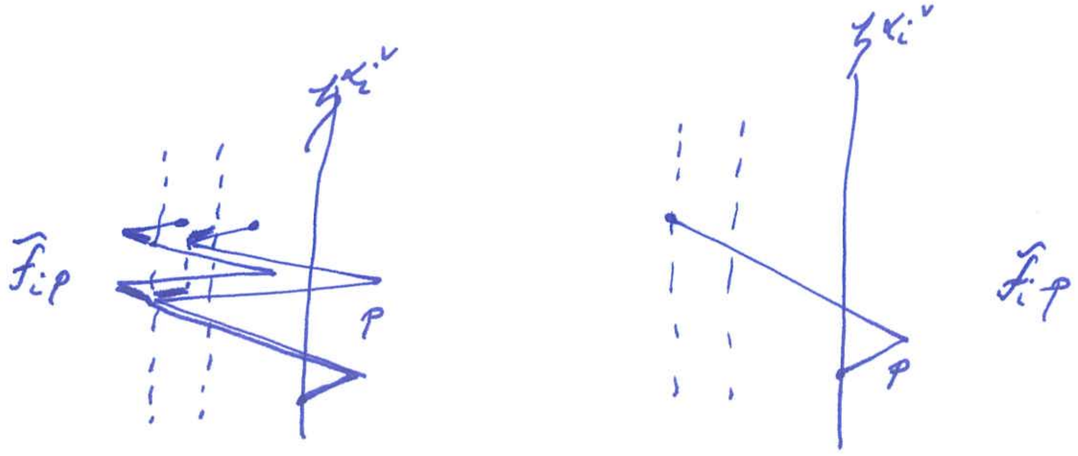
Representation Theory 24.04.2009

Crystals

A *path* is a piecewise linear map $p : [0,1] \to \mathfrak{h}^*_{\mathbb{R}}$ such that $p(0) = 0$ and $p(1) = 1$.

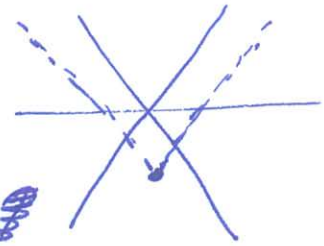
A *crystal* is a collection B of paths which is closed under the action of the root operators $\tilde{e}_i, \tilde{f}_i$



and

$$\tilde{e}_i \tilde{f}_i p = p \text{ if } \tilde{f}_i p \neq 0, \qquad \tilde{f}_i \tilde{e}_i p = p, \text{ if } \tilde{e}_i p \neq 0$$

A *highest weight* p is $p \subseteq C - \rho$



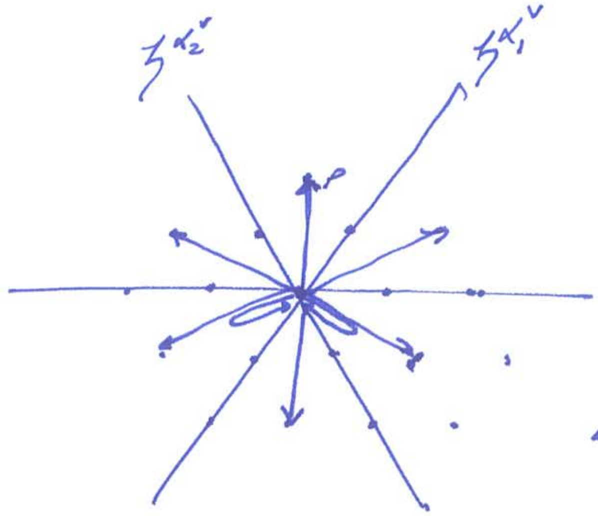
p is highest weight $\Leftrightarrow$ $\tilde{e}_i p = 0$ for all $1 \le i \le n$.

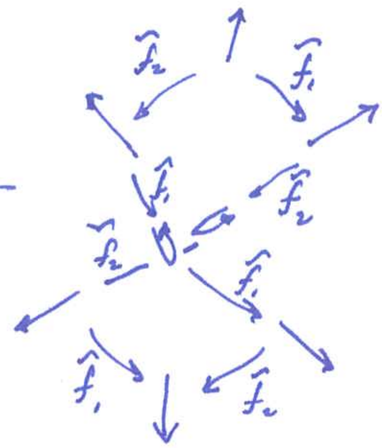
Let $\lambda \in (\mathfrak{h}^*_{\mathbb{Z}})^+$ and p_λ a highest wt path with end λ.

Let

$$B(p_\lambda) = (\text{crystal generated by } p_\lambda).$$

Example





$$\mathrm{char}(B(p_\rho)) = X^{\rho} + X^{s_1\rho} + X^{s_2\rho} + X^{s_1s_2\rho} + X^{s_2s_1\rho} + X^{s_1s_2s_1\rho} + X^{0} + X^{0}.$$

Let B be a crystal and let $p \in B$

The *i-string* of p is

$$\tilde f_i^{\ell} p — \cdots — \tilde f_i^{2} — \tilde f_i p — p — \tilde e_i p — \tilde e_i^{2} p — \cdots — \tilde e_i^{d^+} p$$

where $\tilde e_i^{d^+ +1} p = 0$ and $\tilde f_i^{\ell+1} p = 0$. This is

$$f_i^{\langle\mu,\alpha_i^\vee\rangle} h — \quad \cdots \quad — \tilde f_i h — h$$

where $\mu = wt(h)$, with weight

$$s_i\mu \quad , \quad \ldots, \mu - 2\alpha_i, \mu-\alpha_i, \mu$$

Define an action of W_0 on B by setting

$s_i p$ to be the opposite of p in its i-string.

Then

$$s_i wt(p) = wt(s_i p). \quad \text{for } i = 1, \ldots, n$$

So

$$\mathrm{char}(B) = \mathrm{char}(s_i B) \quad \text{for } i=1,\ldots,n \quad \text{and}$$

$$\mathrm{char}(B) \in \mathbb{C}[X]^{W_0}.$$

<u>Theorem</u> Let B be a crystal. Then

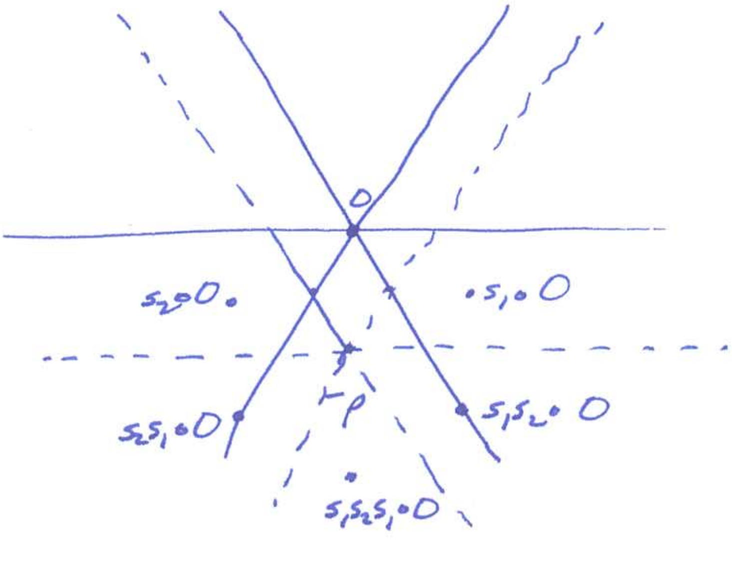
$$\mathrm{char}(B) = \sum_{\substack{p \in B \\ p \subseteq C-\rho}} s_{wt(p)}$$

<u>Proof</u> Define

$$s_\mu = \frac{a_{\mu+\rho}}{a_\rho} \quad \text{for } \mu \in (\mathfrak{h}_{\mathbb{Z}}^*)$$

Define a new action of W_0 on $\mathfrak{h}_{\mathbb{Z}}^*$ by

$$w \circ \mu = w(\mu+\rho) - \rho \quad \text{for } \mu \in \mathfrak{h}_{\mathbb{Z}}^*, \ w \in W_0.$$



Then

$$s_{w\circ\mu} = \frac{a_{w(\mu+\rho)-\rho+\rho}}{a_\rho} = \frac{a_{w(\mu+\rho)}}{a_\rho} = \det(w)\,\frac{a_{\mu+\rho}}{a_\rho}$$

$$= \det(w)\, s_\mu.$$

Let
$$e_0 = \sum_{w \in W_0} \det(w) w \quad \text{so that } e_0(X^\mu) = a_\mu .$$

Then
$$\mathrm{char}(B) = \frac{1}{a_\rho}\, \mathrm{char}(B) a_\rho = \frac{1}{a_\rho}\, \mathrm{char}(B)\, e_0(X^\rho)$$

$$= \frac{1}{a_\rho}\, e_0\big(\mathrm{char}(B) X^\rho\big) = \frac{1}{a_\rho} \sum_{p \in B} e_0\big(X^{wt(p)+\rho}\big)$$

$$= \sum_{p \in B} s_{wt(p)}$$

The equation (*) can cause some cancellation in this sum.

Let $p \in B$ such that p is not highest weight. Let i be minimal such that p leaves $C - \rho$ by crossing $\mathfrak{z}^{\alpha_i + \delta}$. Define $s_i \circ p$ to be the element of the $i$string of p such that

$$wt(s_i \circ p) = s_i \circ wt(p)$$

$$t - \tilde{e}_i t - \tilde{e}_i^2 t - \cdots - \tilde{f}_i^2 h - \tilde{f}_i h - h$$

Then

$$s_{wt(s_i \circ p)} = \det(s_i)\, s_{wt(p)} = -s_{wt(p)}$$

and $s_{wt(s_i \circ p)} + s_{wt(p)} = 0$.

Note that

$s_i \circ p$ leaves $C-\rho$ at the same place that p leaves $C-\rho$. Thus

$$\mathrm{char}(B) = \sum_{\substack{p \in B \\ p \subseteq C-\rho}} s_{\mathrm{wt}(p)}.$$

<u>Theorem</u> $\mathrm{char}(B(\lambda)) = s_\lambda$.