Representation Theory 22.04.2009

Symmetric functions

Let $(W_0, \mathfrak{h}^*_{\mathbb{Z}})$ be a Weyl group,

$\{s_\alpha \mid \alpha \in R^+\}$ the reflections in W_0.

Let

$$X = \{X^\mu \mid \mu \in \mathfrak{h}^*_{\mathbb{Z}}\} \quad \text{with} \quad X^\mu X^\nu = X^{\mu+\nu}.$$

and

$$\mathbb{C}[X] = \text{span}\{X^\mu \mid \mu \in \mathfrak{h}^*_{\mathbb{Z}}\}$$

The ring of symmetric functions is

$$\mathbb{C}[X]^{W_0} = \{f \in \mathbb{C}[X] \mid wf = f, \text{ for all } w \in W_0\}$$

and the vectorspace of determinant symmetric functions

$$\mathbb{C}[X]^{\det} = \{f \in \mathbb{C}[X] \mid wf = \det(w) f, \text{ for all } w \in W_0\}.$$

Let

$$\mathbb{1}_0 = \sum_{w \in W_0} w \quad \text{and} \quad \varepsilon_0 = \sum_{w \in W_0} \det(w^{-1}) w$$

Then

$$\mathbb{C}[X]^{W_0} = \mathbb{1}_0 \mathbb{C}[X] \quad \text{and} \quad \mathbb{C}[X]^{\det} = \varepsilon_0 \mathbb{C}[X].$$

<u>Example</u> (Type GL_3)

$$\mathfrak{h}_{\mathbb{Z}}^* = \text{span}\{\varepsilon_1, \varepsilon_2, \varepsilon_3\} \quad \text{and} \quad W_0 = S_3$$

$$\mathbb{C}[X] = \text{span}\{X^\mu \mid \mu = \mu_1\varepsilon_1 + \mu_2\varepsilon_2 + \mu_3\varepsilon_3 \in \mathfrak{h}_{\mathbb{Z}}^*\} \quad \text{with}$$

$$X^\mu = (X^{\varepsilon_1})^{\mu_1}(X^{\varepsilon_2})^{\mu_2}(X^{\varepsilon_3})^{\mu_3} = x_1^{\mu_1}x_2^{\mu_2}x_3^{\mu_3} \quad \text{if} \quad x_i = X^{\varepsilon_i}.$$

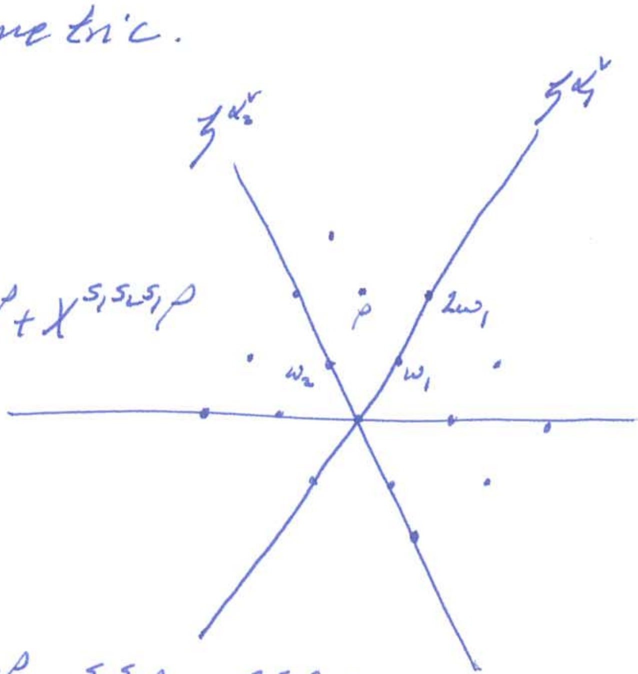
Then

$$m_{(1,1,-2)} = x_1x_2x_3^{-2} + x_1x_2^{-2}x_3 + x_1^{-2}x_2x_3 \quad \text{is symmetric}$$

$$a_{(1,1,-2)} = x_1x_2x_3^{-2} - x_2x_1x_3^{-2} - x_1x_3x_2^{-2} - x_3x_2x_1^{-2} + x_2x_3x_1^{-2} + x_3x_1x_2^{-2}$$

is determinant symmetric.

<u>Example</u> (Type SL_3)



$$m_\rho = X^\rho + X^{s_1\rho} + X^{s_2\rho} + X^{s_1s_2\rho} + X^{s_2s_1\rho} + X^{s_1s_2s_1\rho}$$

$$m_{2\omega_1} = X^{2\omega_1} + X^{2\omega_2 - 2\omega_1} + X^{-2\omega_2}$$

are symmetric

and

$$a_\rho = X^\rho - X^{s_1\rho} - X^{s_2\rho} + X^{s_1s_2\rho} + X^{s_2s_1\rho} - X^{s_1s_2s_1\rho}$$

$$a_{2\omega_1} = X^{2\omega_1} - X^{2\omega_1} - X^{2\omega_2 - 2\omega_1} + X^{2\omega_1 - 2\omega_1} - X^{-2\omega_2} + X^{-2\omega_2}$$

are determinant symmetric.

Bases

$$\mathbb{C}[X]^{W_0} \qquad \mathbb{C}[X]^{\det}$$

$$m_\lambda = \varepsilon_0 X^\lambda$$

$$a_{\lambda+\rho} = \varepsilon_0 X^{\lambda+\rho}$$

The orbit sums or monomial symmetric functions are

$$m_\lambda = \varepsilon_0 X^\lambda = \sum_{\gamma \in W_0\lambda} X^\gamma$$

and

$$a_\mu = \varepsilon_0 X^\mu = \sum_{w \in W_0} \det(w^{-1}) X^{w\mu}.$$

If $\mu \in \mathfrak{h}^{\alpha^\vee}$ then $s_\alpha \mu = \mu$ and

$$~~-a_\mu = s_\alpha a_\mu~~ \quad a_\mu = a_{s_\alpha \mu} = s_\alpha a_\mu = -a_\mu$$

so that $a_\mu = 0$. So

$\{m_\lambda \mid \lambda \in (\mathfrak{h}^*_{\mathbb{Z}})^+\}$ is a basis of $\mathbb{C}[X]^{W_0}$

$\{a_\mu \mid \mu \in (\mathfrak{h}^*_{\mathbb{Z}})^{++}\} = \{a_{\lambda+\rho} \mid \lambda \in (\mathfrak{h}^*_{\mathbb{Z}})^+\}$ is a basis of $\mathbb{C}[X]^{\det}$

Recall

$$\begin{array}{ccc} (\mathfrak{h}^*_{\mathbb{Z}})^* & \xrightarrow{\sim} & (\mathfrak{h}^*_{\mathbb{Z}})^{++} \\ \lambda & \longmapsto & \lambda+\rho. \end{array}$$

where ρ is the minimal length element of $(\mathfrak{h}^*_{\mathbb{Z}})^{++}$

Weyl characters and Weyl denominators

$$\mathbb{C}[X]^{W_0} \longrightarrow \mathbb{C}[X]^{\det}$$
$$f \longmapsto a_\rho f$$

Note: $w(a_\rho f) = (w a_\rho)(w f) = \det(w)\, a_\rho f$, if $f \in \mathbb{C}[X]^{W_0}$

Note: This is a $\mathbb{C}[X]^{W_0}$-module homomorphism.

Claim This is a $\mathbb{C}[X]^{W_0}$-module isomorphism.

Proof Let $g = \sum_{\mu \in \mathfrak{h}_{\mathbb{Z}}^*} g_\mu X^\mu \in \mathbb{C}[X]^{\det}$.

Then
$$g = \tfrac{1}{2}(g+g) = \tfrac{1}{2}(g - s_\alpha g) = \tfrac{1}{2} \sum_{\mu \in \mathfrak{h}_{\mathbb{Z}}^*} g_\mu (X^\mu - X^{s_\alpha \mu})$$

Now
$$X^\mu - X^{s_\alpha \mu} = X^\mu - X^{\mu - \langle \mu, \alpha^\vee \rangle \alpha} = X^\mu (1 - X^{-\langle \mu, \alpha^\vee \rangle \alpha})$$

$$= X^\mu (1 - X^{-\alpha})(1 + X^{-\alpha} + X^{-2\alpha} + \cdots + X^{-(\langle \mu, \alpha^\vee \rangle + 1)\alpha})$$

So $X^\mu - X^{s_\alpha \mu}$ is divisible by $(1 - X^{-\alpha})$

and g is divisible by $1 - X^{-\alpha}$.

So
g is divisible by $\prod_{\alpha \in R^+} (1 - X^{-\alpha})$.

In particular
$$a_\rho = X^\rho \Big(\prod_{\alpha \in R^+} (1 - X^{-\alpha}) \Big).$$

The Weyl character is

$$s_\lambda = \frac{a_{\lambda+\rho}}{a_\rho} \quad \text{i.e.} \quad \begin{array}{ccc} \mathbb{C}[X]^{W_0} & \longrightarrow & \mathbb{C}[X]^{\det} \\ s_\lambda & \longleftrightarrow & a_{\lambda+\rho} \end{array}$$

Example (Type GL_n)

$$\mathfrak{h}_\mathbb{Z}^* = \sum_{i=1}^{n} \mathbb{Z}\varepsilon_i$$

$$C = \{ \mu = \mu_1\varepsilon_1 + \cdots + \mu_n\varepsilon_n \in \mathfrak{h}_\mathbb{R}^* \mid \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \}$$

$$(\mathfrak{h}_\mathbb{Z}^*)^+ = \{ \mu = \mu_1\varepsilon_1 + \cdots + \mu_n\varepsilon_n \in \mathfrak{h}_\mathbb{Z}^* \mid \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \}$$

$$(\mathfrak{h}_\mathbb{Z}^*)^{++} = \{ \mu \in \mathfrak{h}_\mathbb{Z}^* \mid \mu_1 > \cdots > \mu_n \}$$

$$\begin{array}{ccc} (\mathfrak{h}_\mathbb{Z}^*)^+ & \longrightarrow & (\mathfrak{h}_\mathbb{Z}^*)^{++} \\ \mu & \longmapsto & \mu+\rho \end{array} \quad \text{where } \rho = (n-1)\varepsilon_1 + \cdots + 2\varepsilon_{n-2} + \varepsilon_{n-1}$$

and

$$a_\rho = \sum_{w \in S_n} \det(w^{-1})\, w(x_1^{n-1} \cdots x_{n-2}^2 x_{n-1})$$

$$= \det \begin{pmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & x_2 & 1 \\ \vdots & & \ddots & & \\ x_n^{n-1} & x_n^{n-2} & \cdots & x_n & 1 \end{pmatrix}$$

$$= \prod_{1 \leq i < j \leq n} (x_i - x_j) = (x_1^{n-1} \cdots x_{n-2}^2 x_{n-1}) \prod_{1 \leq i < j \leq n} (1 - x_j x_i^{-1}).$$

is the Vandermonde determinant.