Dual vector spaces

A *vector space* V is an R-module with a basis.

The *dual vector space* is $V^* = \mathrm{Hom}(V, R)$. Write

$$\langle \,,\, \rangle : V^* \times V \longrightarrow R \quad \text{and} \quad \langle \mu, \lambda^\vee \rangle = \mu(\lambda^\vee)$$

for $\mu \in V^*$, $\lambda^\vee \in V$.

Let $G = GL(\mathfrak{h}^*)$ so that G acts on $\mathfrak{h}^*$.

Define an action of G on $\mathfrak{h}$ by

$$\langle \mu, g\lambda^\vee \rangle = \langle g^{-1}\mu, \lambda^\vee \rangle .$$

Let $\omega_1, \dots, \omega_n$ be a basis of $\mathfrak{h}^*$.

The *dual basis* in $\mathfrak{h}$ is $\{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ such that

$$\langle \omega_j, \alpha_i^\vee \rangle = \delta_{ij} .$$

The matrix of the action of g on $\mathfrak{h}$ is

$$g^\vee = (g^t)^{-1} .$$

Reflections

A *reflection* is $s_\alpha \in GL(\mathfrak{h}^*)$ such that

$$s_\alpha \text{ is conjugate to } \begin{pmatrix} \xi & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

with $\xi \in \mathbb{F}$, $\xi \neq 1$.

Then

$$s_\alpha^\vee \in GL(\mathfrak{h}) \text{ is conjugate to } \begin{pmatrix} \xi^{-1} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

Then

$$\mathfrak{h}^* = \mathfrak{h}^{\alpha^\vee} \oplus \mathbb{C}\alpha \quad \text{and} \quad \mathfrak{h} = \mathfrak{h}^\alpha \oplus \mathbb{C}\alpha^\vee$$

where

$$\mathfrak{h}^{\alpha^\vee} = (\mathfrak{h}^*)^{s_\alpha} = \{\mu \in \mathfrak{h}^* \mid s_\alpha \mu = \mu\} = 1 \text{ eigenspace of } s_\alpha$$

$$\mathbb{C}\alpha = \xi\text{-eigenspace of } s_\alpha$$

$$\mathfrak{h}^\alpha = \mathfrak{h}^{s_\alpha^\vee} = \{\lambda^\vee \in \mathfrak{h} \mid s_\alpha^\vee \lambda^\vee = \lambda^\vee\} = 1 \text{ eigenspace of } s_\alpha^\vee$$

$$\mathbb{C}\alpha^\vee = \xi^{-1}\text{-eigenspace of } s_{\alpha^\vee}$$

Then

$$\mathfrak{h}^{\alpha^\vee} = \{\mu \in \mathfrak{h}^* \mid \langle \mu, \alpha^\vee \rangle = 0\} \quad \text{and}$$

$$\mathfrak{h}^\alpha = \{\mu \in \mathfrak{h}^* \mid \langle \lambda^\vee, \alpha \rangle = 0\} \quad \text{and if}$$

α and $\alpha^\vee$ are normalized so that

$$\langle \alpha, \alpha^\vee \rangle = 1 - \xi$$

then

$$s_\alpha \mu = \mu - \langle \mu, \alpha^\vee \rangle \alpha \quad \text{and} \quad s_\alpha^{-1} \lambda^\vee = \lambda^\vee - \langle \lambda^\vee, \alpha \rangle \alpha^\vee.$$

Weyl groups

Let $\mathfrak{h}^*_{\mathbb{Z}}$ be a $\mathbb{Z}$-vector space

$$\mathfrak{h}^*_{\mathbb{Z}} = \mathbb{Z}\text{-span}\{\omega_1, \ldots, \omega_n\}$$

where $\omega_1, \ldots, \omega_n$ is a $\mathbb{Z}$-basis of $\mathfrak{h}^*_{\mathbb{Z}}$. Then

$$\mathfrak{h}^*_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathfrak{h}^*_{\mathbb{Z}} = \mathbb{Q}\text{span}\{\omega_1, \ldots, \omega_n\}$$

$$\mathfrak{h}^*_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Q}} \mathfrak{h}^*_{\mathbb{Z}} = \mathbb{R}\text{-span}\{\omega_1, \ldots, \omega_n\}$$

$$\mathfrak{h}^*_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} \mathfrak{h}^*_{\mathbb{Z}} = \mathbb{C}\text{-span}\{\omega_1, \ldots, \omega_n\}$$

$$\mathfrak{h}^*_{\overline{\mathbb{Q}}} = \overline{\mathbb{Q}} \otimes_{\mathbb{Z}} \mathfrak{h}^*_{\mathbb{Z}} = \overline{\mathbb{Q}}\text{-span}\{\omega_1, \ldots, \omega_n\}$$

A <u>Weyl group</u> is a finite subgroup ~~of~~ W_0 of $GL(\mathfrak{h}^*_{\mathbb{Z}})$ generated by reflections.

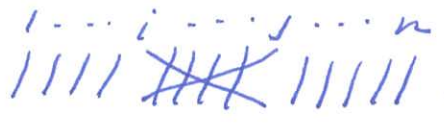
Let R^+ be an index set for the reflections s_α in W_0.

Example 1

$\mathfrak{h}_{\mathbb{Z}}^* = \mathrm{span}\{\varepsilon_1, \ldots, \varepsilon_n\}$ with $W_0 = S_n$

acting by permuting $\varepsilon_1, \ldots, \varepsilon_n$. The reflections in S_n are

$$s_{ij} = s_{\varepsilon_i^\vee - \varepsilon_j^\vee} =$$



$$= \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccc} \quad & & i & & j & \end{array} \\ \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 0 & & 1 & & \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & 1 & \\ & & & 1 & & & & 0 \\ & & & & & & & & 1 \\ & & & & & & & & & \ddots \\ & & & & & & & & & & 1 \end{pmatrix} \end{array}$$

and

$$R^+ = \{(ij) \mid 1 \le i < j \le n\} \quad \text{or} \quad R^+ = \{\varepsilon_i^\vee - \varepsilon_j^\vee \mid 1 \le i < j \le n\}$$

and

$$\mathfrak{h}^{\varepsilon_i^\vee - \varepsilon_j^\vee} = (\mathfrak{h}^*)^{s_{ij}} = \{\mu \in \mathfrak{h}_{\mathbb{R}}^* \mid s_{ij}\mu = \mu\}$$

$$= \{\mu = \mu_1\varepsilon_1 + \cdots + \mu_n\varepsilon_n \mid \langle \mu, \varepsilon_i^\vee - \varepsilon_j^\vee \rangle = 0\}$$

$$= \{\mu = \mu_1\varepsilon_1 + \cdots + \mu_n\varepsilon_n \mid \mu_i = \mu_j\}$$

The arrangement of hyperplanes

$$\mathfrak{h}^{\varepsilon_i^\vee - \varepsilon_j^\vee}, \quad 1 \le i < j \le n \quad \text{in} \quad \mathfrak{h}_{\mathbb{C}}^*$$

is the <u>braid</u> <u>arrangement</u>

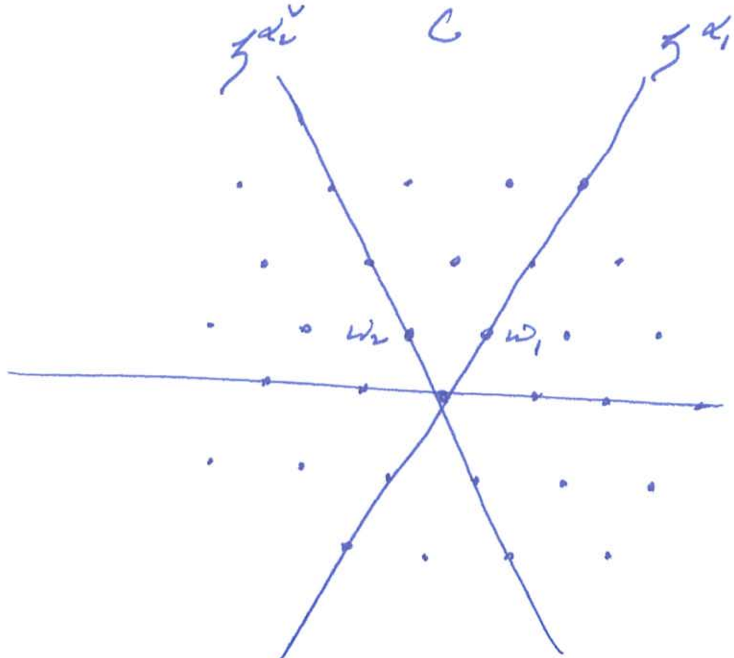
<u>Remark:</u> $\mathrm{Conf}_n(\mathbb{C}^n) = \left(\mathfrak{h}_{\mathbb{C}}^* - \left(\bigcup_{1 \le i < j \le n} \mathfrak{h}^{\varepsilon_i^\vee - \varepsilon_j^\vee}\right)\right)$

has

$$\pi_1(\mathrm{Conf}_n(\mathbb{C}^n)) = \text{braid group } B_n$$

Example (Type SL_3)

$\mathfrak{h}_{\mathbb{Z}}^* = \text{span}\{\omega_1, \omega_2\}$ and $W_0 = \left\langle s_1, s_2 \;\middle|\; \begin{matrix} s_1^2 = s_2^2 = 1, \\ s_1 s_2 s_1 = s_2 s_1 s_2 \end{matrix} \right\rangle$



where s_1 is reflection in $\mathfrak{h}^{\alpha_1^\vee}$

s_2 is reflection in $\mathfrak{h}^{\alpha_2^\vee}$.

Let C be a fundamental chamber for the action of W_0 on $\mathfrak{h}_{\mathbb{R}}^*$.

$$W_0 \xleftrightarrow{1-1} \{\text{chambers in } \mathfrak{h}_{\mathbb{R}}^*\}$$

Let $\bar{C}$ be the closure of C

The dominant integral weights are

$$P^+ = \mathfrak{h}_{\mathbb{Z}}^* \cap \bar{C} \quad \text{and} \quad P^{++} = \mathfrak{h}_{\mathbb{Z}}^* \cap C$$

are the strictly dominant integral weights

There is a bijection $\begin{matrix} P^+ & \longrightarrow & P^{++} \\ \lambda & \longmapsto & \lambda + \rho \end{matrix}$

where ρ is the point of P^{++} closest to 0