Representation Theory 19.05.2009.

The groups U_n, O_n, Sp_n

The unitary group

$$U(n) = \{ g \in GL_n(\mathbb{C}) \mid g\bar{g}^t = 1 \}$$

where $\bar{g} = (\bar{g}_{ij})$ if $g = (g_{ij})$.

The orthogonal group is

$$O_n(\mathbb{C}) = \{ g \in GL_n(\mathbb{C}) \mid gg^t = 1 \}$$

The symplectic group is

$$Sp_{2n}(\mathbb{C}) = \{ g \in GL_n(\mathbb{C}) \mid gJg^t = J \}$$

where

$$J = \left(\begin{array}{c|c} 0 & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \\ \hline \begin{matrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{matrix} & 0 \end{array}\right) \quad \text{or} \quad J = \left(\begin{array}{c|c} 0 & \begin{matrix} 0 & & 1 \\ & ⋰ & \\ 1 & & 0 \end{matrix} \\ \hline \begin{matrix} 0 & & -1 \\ & ⋰ & \\ -1 & & 0 \end{matrix} & 0 \end{array}\right)$$

~~Let V be a vector space over $\mathbb{F}$.~~

~~A symmetric bilinear form on V is a map~~

$$\cancel{\langle\, ,\, \rangle : V \times V \longrightarrow \mathbb{F}}$$

$$\cancel{(v_1, v_2) \longmapsto \langle v_1, v_2 \rangle} \quad \text{~~such that~~}$$

Let V be an $\mathbb{F}$-vector space.

A <u>bilinear form</u> on V is a map $\langle\, ,\rangle : V\times V \to \mathbb{F}$ such that

$$\langle c_1v_1+c_2v_2, v_3\rangle = c_1\langle v_1,v_3\rangle + c_2\langle v_2,v_3\rangle, \text{ and}$$

$$\langle v_1, c_1v_2+c_2v_3\rangle = c_1\langle v_1,v_2\rangle + c_2\langle v_1,v_3\rangle$$

for $v_1,v_2,v_3\in V$, $c_1,c_2,c_3\in\mathbb{F}$. A bilinear form $\langle\, ,\rangle : V\times V\to\mathbb{F}$ is <u>symmetric</u> if

$$\langle v_1,v_2\rangle = \langle v_2,v_1\rangle, \quad \text{for } v_1,v_2\in V,$$

and <u>skew-symmetric</u> if

$$\langle v_1,v_2\rangle = -\langle v_2,v_1\rangle, \quad \text{for } v_1,v_2\in V.$$

The <u>orthogonal group</u> is

$$O_n(\mathbb{F}) = O(V,\langle\ \rangle)$$
$$= \{g\in GL(V) \mid \langle gv_1,gv_2\rangle = \langle v_1,v_2\rangle, \text{ for } v_1,v_2\in V\}$$

The <u>symplectic group</u> is

$$Sp_n(\mathbb{F}) = Sp(V,\langle\ \rangle)$$

$$= \{g\in GL(V) \mid \langle gv_1,gv_2\rangle = \langle v_1,v_2\rangle, \text{ for } v_1,v_2\in V\}$$

Let $\mathbb{F}$ be a field with an involution $\bar{\ }: \mathbb{F} \to \mathbb{F}$.
$z \mapsto \bar{z}$

Let V be a vector space over $\mathbb{F}$.

A <u>sesquilinear form</u> is a map $\langle , \rangle : V \times V \to \mathbb{F}$ such that

$$\langle c_1 v_1 + c_2 v_2, v \rangle = c_1 \langle v_1, v \rangle + c_2 \langle v_2, v \rangle,$$

$$\langle w, a_1 w_1 + a_2 w_2 \rangle = \bar{a}_1 \langle w, w_1 \rangle + \bar{a}_2 \langle w, w_2 \rangle$$

for $v, v_1, v_2, w, w_1, w_2 \in V$ and $a_1, a_2, c_1, c_2 \in \mathbb{F}$.

A <u>Hermitian form</u> on V is a sesquilinear form $\langle , \rangle : V \times V \to \mathbb{F}$ such that

$$\langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle} \quad \text{for} \quad v_1, v_2 \in V.$$

A <u>positive</u> <u>Hermitian form</u> on V is a Hermitian form such that

$$\langle v, v \rangle \in \mathbb{R}_{\geq 0}, \text{ for all } v \in V$$

Let $\langle \ \rangle : V \times V$ be a Hermitian form on V.

The <u>unitary group</u>

$$U_n = \left\{ g \in GL(V) \;\middle|\; \begin{array}{l} \langle v_1, v_2 \rangle = \langle g v_1, g v_2 \rangle \\ \text{for } v_1, v_2 \in V \end{array} \right\}$$

Let

$$(V\otimes V)^* = \{\text{bilinear forms on } V\}$$

$$S^2(V)^* = \{\text{symmetric bilinear forms on } V\}$$

$$\Lambda^2(V)^* = \{\text{skew-symmetric bilinear forms on } V\}.$$

The symmetric group $S_2 = \{1, s\}$ with $s^2=1$ acts on $(V\otimes V)^*$ by

$$(s\cdot\langle\ \rangle)(v_1,v_2) = \langle v_2, v_1\rangle, \quad \text{for } v_1, v_2 \in V.$$

Then

$$S^2(V)^* = \{\langle\ \rangle \in (V\otimes V)^* \mid s\cdot\langle\ \rangle = \langle , \rangle\}$$

$$\Lambda^2(V)^* = \{\langle\ \rangle \in (V\otimes V)^* \mid s\cdot\langle\ \rangle = -\langle , \rangle\}.$$

The group $GL(V)$ acts on $(V\otimes V)^*$ by

$$(g\cdot\langle\ \rangle)(v_1,v_2) = \langle g^{-1}v_1, g^{-1}v_2\rangle,$$

for $g\in GL(V)$, $\langle\ \rangle\in(V\otimes V)^*$, $v_1,v_2\in V$. The $GL(V)$-action on $(V\otimes V)^*$ commutes with the S_2-action on $(V\otimes V)^*$,

$$gs\cdot\langle\ \rangle = sg\cdot\langle\ \rangle, \quad \text{for } g\in GL(V),\ \langle\ \rangle\in(V\otimes V)^*$$

and

$$(V\otimes V)^* = S^2(V)^* \oplus \Lambda^2(V)^*, \quad \text{as } (GL(V), S_2)\text{-bimodul}$$

A choice of basis $b_1, \dots, b_n$ of V provides a bijection

$$(V \otimes V)^* \longrightarrow M_n(\mathbb{F})$$
$$\langle\, , \,\rangle \longmapsto A = (\langle b_i, b_j \rangle)_{1 \le i,j \le n}$$

and this bijection identifies

$$S^2(V)^* \text{ with } \mathrm{Sym}_n(\mathbb{F}) = \{A \in M_n(\mathbb{F}) \mid A = A^t\}$$
$$\Lambda^2(V)^* \text{ with } \mathrm{Skew}_n(\mathbb{F}) = \{A \in M_n(\mathbb{F}) \mid A = -A^t\}$$

The S_2-action and the $GL(V)$-action on $(V \otimes V)^*$ become the actions of S_2 and $GL(V)$ on $M_n(\mathbb{F})$ given by

$$s \cdot A = A^t \quad \text{and} \quad g \cdot A = g^{-1} A (g^{-1})^t$$

for $a \in M_n(\mathbb{F})$ and $g \in GL_n(\mathbb{F})$.