Representation Theory class 11.03.2009

Let A be an algebra

Let M and N be simple A-modules

Let $\varphi : M \to N$ an A-module homomorphism.

i.e. $\varphi a_{M} = a_N \varphi$ for all $a \in A$; i.e. $\varphi \in Hom_A(M,N)$.

Then $\ker\varphi$ and $\operatorname{im}\varphi$ are submodules of M and N respectively. So

$$\ker\varphi = 0 \text{ or } M \text{ and } \operatorname{im}\varphi = 0 \text{ or } N.$$

Then $\varphi = 0$ or φ is an isomorphism.

Let $\varphi : M \to M$ be an A-module homomorphism. Let λ be an eigenvalue of φ. Then

$$\varphi - \lambda \in End_A(M).$$

So $\varphi - \lambda = 0$ or $\varphi - \lambda$ is ~~in~~vertible.

Since $\det(\varphi - \lambda) = 0$, $\varphi - \lambda$ is not invertible.

So $\varphi = \lambda \cdot Id$.

<u>Schur's Lemma</u> Let M and N be simple modules.

If $M \not\cong N$ then $Hom_A(M,N) = 0$.

If $M \cong N$ then $End_A(M) = \mathbb{C} \cdot Id_M$.

Let A be an algebra.

A *trace* on A is a linear map $\vec{t}: A \to \mathbb{C}$ such that

$$\vec{t}(a_1 a_2) = \vec{t}(a_2 a_1) \quad \text{for all } a_1, a_2 \in A.$$

Let M be an A-module. Then

$$\begin{aligned} A &\to End(M) \\ a &\mapsto a_M \end{aligned} \quad \text{is a homomorphism}$$

and $$\begin{aligned} \chi_M : A &\to \mathbb{C} \\ a &\mapsto Tr(a_M) \end{aligned} \quad \text{where } Tr(a_M) = \sum_{b \in B} ab|_b$$

with B a basis of M.

Then χ_M is a trace on A.

The *regular representation of A* is the vector space A with A acting ~~as~~ by left multiplication. Let

$$\begin{aligned} \vec{t}_A : A &\to \mathbb{C} \\ a &\mapsto Tr(a_A) \end{aligned} \quad \text{be the trace of the regular representation.}$$

Example Let G be a group. ~~algebra~~ The *group algebra* of G is the vector space

$\mathbb{C}G = \text{span}\{g \mid g \in G\}$ with product given by ~~a~~ the product in ~~a~~ G.

Then $\vec{t}_{\mathbb{C}G} : \mathbb{C}G \to \mathbb{C}$ is given by

$$\vec{t}_{\mathbb{C}G}(g) = \sum_{h \in G} gh|_h = \sum_{h \in G} \delta_{g1}$$

$$= \begin{cases} 0, & \text{if } g \neq 1, \\ |G|, & \text{if } g = 1, \end{cases}$$

Note that

$\vec{t}_{\mathbb{C}G} = |G| \vec{t}_1$ where $\vec{t}_1 : \mathbb{C}G \to \mathbb{C}$ is given by

$$\vec{t}_1(g) = \begin{cases} 0, & \text{if } g \neq 1 \\ 1, & \text{if } g = 1 \end{cases}.$$

Let $\vec{t} : A \to \mathbb{C}$ be a trace on A. Define

$\langle\ \rangle : A \otimes A \to \mathbb{C}$ by

$$\langle a_1, a_2 \rangle = \vec{t}(a_1 a_2).$$

Then $\langle a_1, a_2 \rangle = \langle a_2, a_1 \rangle$ for $a_1, a_2 \in A$, so that

$\langle , \rangle$ is a symmetric bilinear form.

The <u>Radical of $\langle\ \rangle$</u> is

$$\mathrm{Rad}(\langle\ \rangle) = \{ a \in A \mid \langle a, a' \rangle = 0 \text{ for all } a' \in A \}.$$

The <u>radical</u> of $\vec{t}$ is $\mathrm{Rad}(\langle , \rangle)$ where $\langle a_1, a_2 \rangle = \vec{t}(a_1 a_2)$.

The form $\langle\ \rangle$ (or the trace $\vec{t}$) is <u>nondegenerate</u> if $\mathrm{Rad}(\langle\ \rangle) = 0$.

Let $\{b_1, \ldots, b_\ell\}$ be a basis of A.

The <u>Gram matrix</u> of $\langle\ \rangle$ is

$$g_{\langle,\rangle} = (\langle b_i, b_j \rangle).$$

<u>Proposition</u>

$$\mathrm{Rad}(\langle\ \rangle) = 0 \iff g_{\langle,\rangle} \text{ is invertible}$$

$\iff$ The dual basis $\{b_1^*, \ldots, b_\ell^*\}$ exists.

<u>Theorem</u> Let A be an algebra such that $\vec{t}_A$ is well defined and nondegenerate.

Let $B = \{b\}$ be a basis of A and let $\{b^*\}$ be the dual basis of A with respect to $\langle\ \rangle$, where $\langle a_1, a_2 \rangle = \vec{t}_A(a_1 a_2)$.

Let M and N be A-modules and let

$\varphi : M \to N$ be a linear transformation.

Let

$$[\varphi] = \sum_{b \in B} b \varphi b^* \quad \text{so that } [\varphi] : M \to N.$$

Then

$[\varphi]$ is an A-module homomorphism, i.e.

$$a[\varphi] = [\varphi]a \quad \text{for all } a \in A.$$

(b) Let $a \in A$ and let

$$[a] = \sum_{b \in B}' b a b^*. \quad \text{Then } [a] \in Z(A).$$

(c) Let M be an A-module.

Let N be a submodule of M. Let $\{n_1, \dots, n_r\}$ be a basis of N and $\{n_1, \dots, n_r, m_1, \dots, m_s\}$ a basis of M. Let

$$\begin{aligned} \varphi : M &\longrightarrow N \\ n_i &\longmapsto n_i \quad \text{for } i = 1, \dots, r \\ m_j &\longmapsto 0, \quad \text{for } j = 1, \dots, s. \end{aligned}$$

Let
$$P = (1 - [\varphi]) M.$$
Then P is a submodule of M and

$$M = N \oplus P.$$