Representation Theory class 10.03.2009 (1)

An <u>algebra</u> is a vector space A with a product $A \otimes A \to A$ such that

(1) the product is associative

(2) there is an identity

Note: $A \otimes A \to A$ means

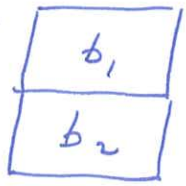
$$(c_1 a_1 + c_2 a_2) a_3 = c_1 (a_1 a_3) + c_2 (a_2 a_3) \quad \text{and}$$

$$a_1 (c_3 a_2 + c_4 a_3) = c_3 (a_1 a_2) + c_4 (a_1 a_3).$$

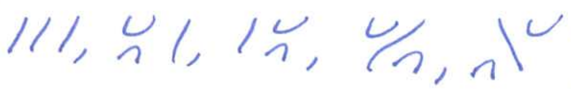
<u>Example</u> The Temperley-Lieb algebra TL_k is

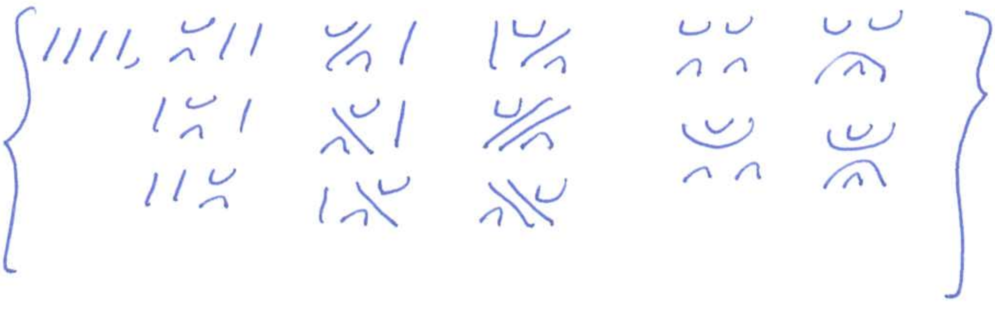
$$TL_k = \text{span}\left\{ \begin{array}{l} \text{noncrossing diagrams with} \\ k \text{ top dots and } k \text{ bottom dots} \end{array} \right\}$$

with product

$$b_1 b_2 = (q + q^{-1})^{\#\text{ of internal loops}}$$ 

$TL_1 = \text{span}\{ | \}$, $TL_2 = \text{span}\{ ||,$  $\}$

$TL_3 = \{$  $\}$ and

$TL_4 =$ 

<u>Theorem</u> TL_k is presented by generators

$e_1, \ldots, e_{k-1}$,

$$e_i = |||| \overset{i\ \ i+1}{\underset{\cap}{\cup}} ||||, \qquad 1 \le i \le k-1,$$

and relations

$$e_i^2 = (q+q^{-1})e_i, \quad e_i e_{i\pm1} e_i = e_i.$$

<u>Proof</u> To show:

(a) Generators A can be written in terms of Generators B.

(b) Relations A can be derived from Relations B.

(c) Generators B can be ~~derive~~ written in terms of Generators A

(d) Relations B can be derived from relations A.

<u>Example</u> $M_2(\mathbb{C}) \oplus M_1(\mathbb{C}) = \left\{ \begin{pmatrix} \boxed{\begin{matrix} * & * \\ * & * \end{matrix}} & \\ & \boxed{*} \end{pmatrix} \right\}$

which has basis

$$\{ E_{11}^{\phi}, E_{12}^{\phi}, E_{21}^{\phi}, E_{22}^{\phi}, E_{11}^{\square\square} \}$$

where E_{ij}^{λ} = has a 1 in the ij entry of the λ^{th} block and 0's elsewhere.

More generally, $\bigoplus_{\lambda} M_{d_\lambda}(\mathbb{C})$ has basis

$$\{ E_{ij}^{\lambda} \mid \lambda \in \hat{A},\ 1 \le i,j \le d_\lambda \} \text{ with } E_{ij}^{\lambda} E_{rs}^{\mu} = \delta_{\lambda\mu}\delta_{jr} E_{is}^{\lambda}.$$

<u>Theorem</u> Most algebras are split semisimple.

i.e. Most algebras are isomorphic to

$$\bigoplus_{\lambda\in\hat{A}} M_{d_\lambda}(\mathbb{C}) \text{ for some } \hat{A} \text{ and } d_\lambda.$$

<u>Example</u>: TL_3 acts on $M = \mathrm{span}\{\cup\cdot, \dot{\cup}, \cdot\cup\}$

Use the basis

$$n_1 = \cup\cdot,\quad n_2 = \cdot\cup,\quad n_3 = ((q+q^{-1})+1)\,\dot{\cup} - \cup\cdot - \cdot\cup$$

Then

$$\overset{\cup}{\cap}|\, n_1 = (q+q^{-1})n_1$$

$$\overset{\cup}{\cap}|\, n_2 = n_1$$

$$\overset{\cup}{\cap}|\, n_3 = \big(((q+q^{-1})+1) - (q+q^{-1}) - 1\big)\ \cup\cdot = 0.$$

and

$$|\overset{\cup}{\cap}\, n_1 = n_2$$

$$|\overset{\cup}{\cap}\, n_2 = (q+q^{-1})n_1$$

$$|\overset{\cup}{\cap}\, n_3 = \big(((q+q^{-1})+1) - 1 - (q+q^{-1})\big)\ \cdot\cup = 0$$

So

$\overset{\cup}{\cap}|$ acts by $\left(\begin{array}{cc|c} q+q^{-1} & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$ and $|\overset{\cup}{\cap}$ acts by $\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & q+q^{-1} & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$

$|||$ acts by $\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$

so that

$$TL_3 \cong M_2(\mathbb{C}) \oplus M_1(\mathbb{C})$$

 $\longmapsto \left(\begin{array}{cc|c} q+q^{-1} & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right)$

 $\longmapsto \left(\begin{array}{cc|c} 0 & 0 & \\ q+q^{-1} & 1 & \\ \hline & & 0 \end{array}\right)$

$||| \longmapsto \left(\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & \\ \hline & & 1 \end{array}\right).$

Let A be an algebra, M an A-module.

The <u>commutant</u> or <u>centralizer</u> <u>algebra</u> is

$$\mathrm{End}_A(M) = \{\varphi \in \mathrm{End}(M) \mid a_M \varphi = \varphi a_M \text{ for } a \in A\}$$

where

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{End}(M) \\ a & \longmapsto & a_M \end{array}$$ is the algebra homomorphism

corresponding to the action of A on M.

~~Theorem (Schur's lemma). If M is~~

Let M and N be simple modules and

$\varphi: M \to N$ an A-module homomorphism.

(i.e. $\varphi a_M = a_N \varphi$ for $a \in A$). Then

$\ker\varphi$ and $\mathrm{im}\,\varphi$ are submodules

of M and N respectively. So $\ker\varphi = 0$ or M

and $\mathrm{im}\,\varphi = 0$ or N. So

$\varphi = 0$ or φ is a bijection (and $M \cong N$).

Let λ be an eigenvalue of φ. Then

$\varphi - \lambda \in End_A(M)$. So $\varphi - \lambda = 0$ or $\varphi - \lambda$ is invertible.

Since $\det(\varphi - \lambda) = 0$, $\varphi - \lambda$ is not invertible.

So $\varphi = \lambda \cdot Id$.

Schur's lemma If M is a simple module, then

$$End_A(M) = \mathbb{C} \cdot Id_M.$$