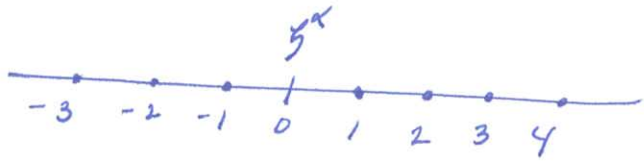
Representation Theory class 08.04.2009

For sl_3 crystals the pictures

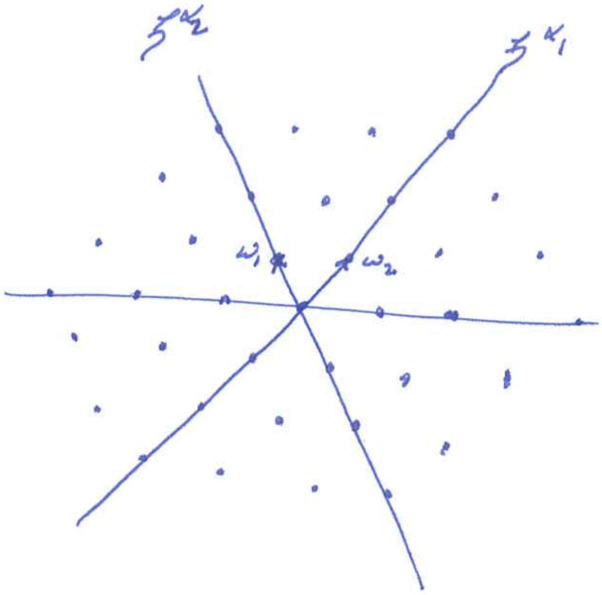
sl_2



and

$B(\square)$ with $\tilde{e}$ and $\tilde{f}$, are replaced by

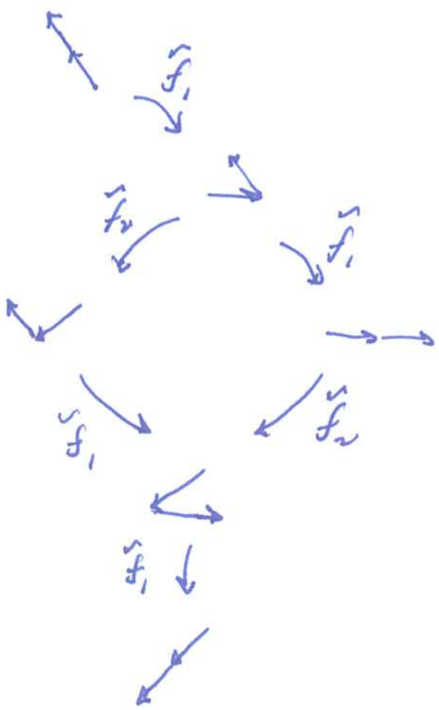
sl_3

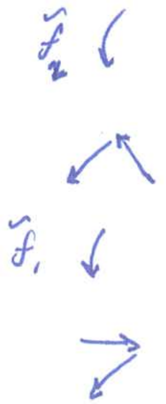


with operators $\tilde{e}_1, \tilde{f}_1$ and $\tilde{e}_2, \tilde{f}_2$

$B(\square) = \{ \nwarrow, \rightarrow, \swarrow \}$. with $char(B(\square)) = x_1 + x_2 + x_3$.

Then $B(\square) \otimes B(\square)$





$= B(\square\square) \sqcup B(\begin{smallmatrix}\square\\\square\end{smallmatrix})$.

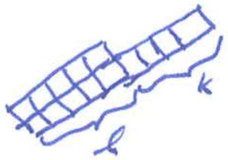
The points of the positive/dominant chamber

$$P^+ = \{ k\omega_1 + l\omega_2 \mid k, l \in \mathbb{Z}_{\geq 0} \}$$

are in bijection with partitions with ≤ 2 rows

$$P^+ \longrightarrow \{\text{partitions with } \leq 2 \text{ rows}\}$$

$$k\omega_1 + l\omega_2 \longmapsto$$



We have

$$\mathrm{char}(B(\square) \otimes B(\square)) = (x_1 + x_2 + x_3)^2$$

$$= (x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2)$$

$$+ (x_1x_2 + x_1x_3 + x_2x_3) \qquad \text{with}$$

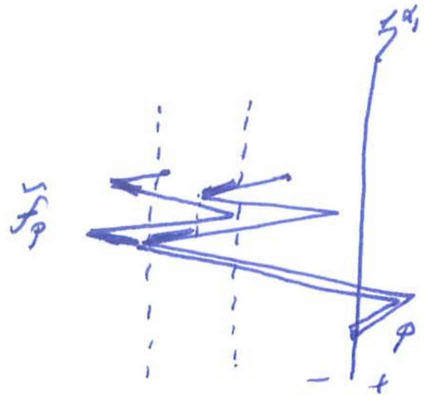
$$\mathrm{char}(B(\square\square)) = x_1^2 + x_2x_1 + x_3x_1 + x_2^2 + x_3x_2 + x_3^2$$

$$= \sum_{1 \leq i \leq j \leq 3} x_ix_j \qquad \text{and}$$

$$\mathrm{char}(B(\begin{smallmatrix}\square\\\square\end{smallmatrix})) = x_1x_2 + x_1x_3 + x_2x_3 = \sum_{1 \leq i < j \leq 3}$$

An *$\mathfrak{sl}_3$-crystal* is a collection of paths closed under the root operators $\tilde{e}_1, \tilde{f}_1, \tilde{e}_2, \tilde{f}_2$.

The root operators $\tilde{e}_1, \tilde{f}_1$ act like the $\mathfrak{sl}_2$-crystal operators $\tilde{e}, \tilde{f}$ in the $(\zeta^{\alpha_1})^\perp$ projection



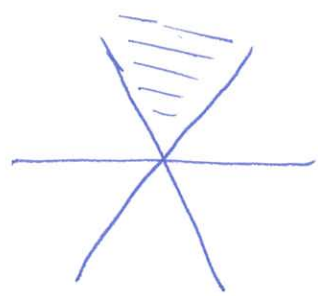
and

has $\tilde{f}_1 p = 0$

and $\tilde{e}_2, \tilde{f}_2$ act like $\tilde{e}, \tilde{f}$ in the $(\zeta^{\alpha_2})^\perp$ projection.

A *highest weight path* is a path p contained in

$$C = \bigcap \begin{pmatrix}\text{positive} \\ \text{half spaces}\end{pmatrix} =$$



A path p is highest weight if and only if

$$\tilde{e}_1 p = 0 \quad \text{and} \quad \tilde{e}_2 p = 0.$$

The *crystal graph* has edges labeled $\tilde{f}_1, \tilde{f}_2$.

The crystal graph is *irreducible* if the crystal graph is connected.

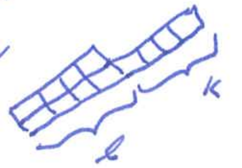
Theorem

(a) The irreducible $\mathfrak{sl}_3$-crystals are indexed by the points in

$P^+ = \{k\omega_1 + l\omega_2 \mid k, l \in \mathbb{Z}_{\geq 0}\}$

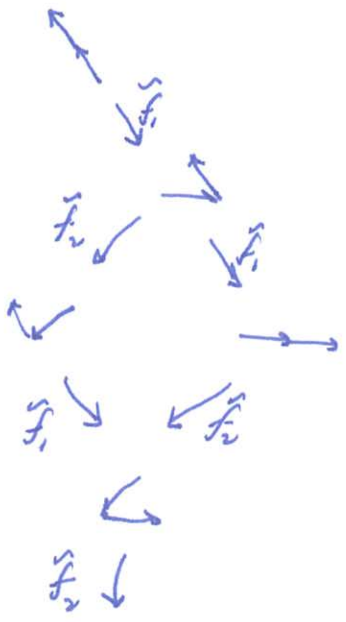
(b) Every $\mathfrak{sl}_3$-crystal is a disjoint union of irreducible crystals.

(c) Each irreducible crystal B has a unique highest weight path p and

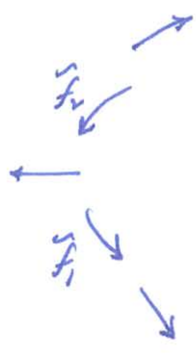
$B \simeq B($  $)$ if p ends at $k\omega_1 + l\omega_2$.

Example We had

$B(\square\square) =$

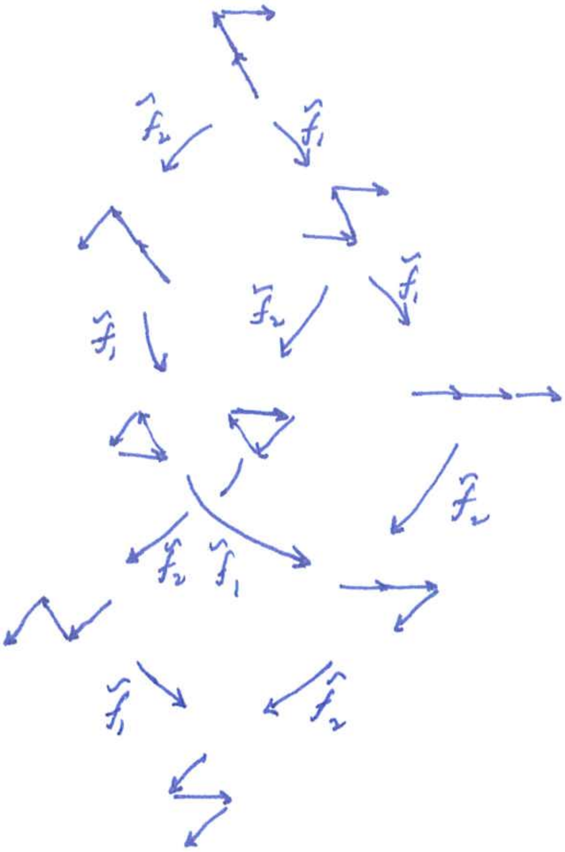


$B\left(\begin{smallmatrix}\square\\ \square\end{smallmatrix}\right) =$

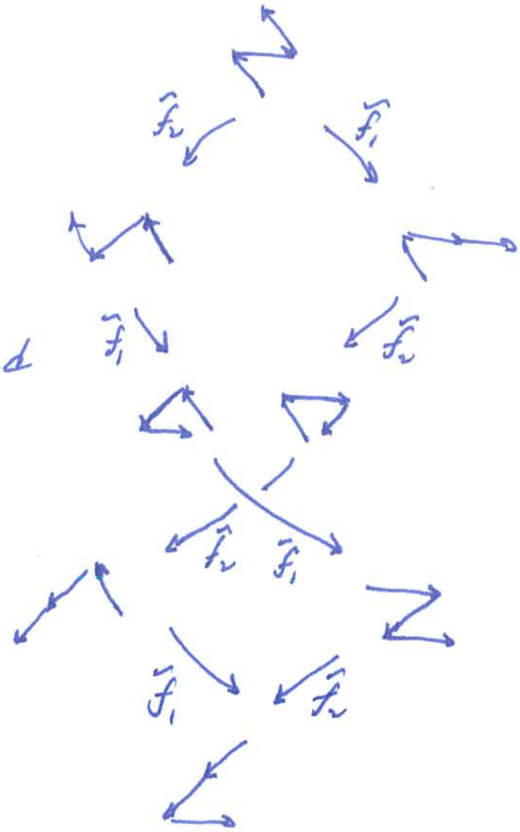


and $B(\square) \otimes B(\square) \simeq B(\square\square) \sqcup B\left(\begin{smallmatrix}\square\\ \square\end{smallmatrix}\right)$.

Then $B(\diamond\!\!\diamond)\otimes B(\diamond)$ contains

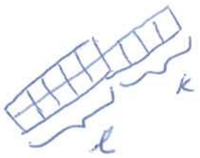


and



is contained in $B(\diamond\!\!\diamond)\otimes B(\diamond)$

A <u>column strict</u> <u>tableau</u> of shape

$\lambda =$  is a filling of the boxes of λ

from $\{1,2,3\}$ such that

(a) rows weakly increase (left to right)

(b) columns strictly increase (right to left)

Let $\quad p_1 = \nwarrow, \quad p_2 = \longrightarrow, \quad p_3 = \searrow$

$$p = \underbrace{p_1 p_1 \cdots p_1}_{k+l}\ \underbrace{p_2 p_2 \cdots p_2}_{k}$$

There is a bijection from

$B(p)$ = the irreducible crystal with highest weight path p

&

$B(\lambda) = \left\{ \begin{array}{l} \text{column strict tableau of shape } \lambda \\ \text{filled from } \{1, 2, 3\} \end{array} \right\}$