620-619 Representation Theory
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University of Melbourne
Mathematics Department

Homework Due 7 April 2009

1. Classify and construct the finite dimensional simple modules for $U_q\mathfrak{sl}_2$, where $U_q\mathfrak{sl}_2$ is the algebra generated by $E, F, K^{\pm 1}$, with relations
$$KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \quad \text{and} \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

2. Define the symmetric group (via permutations).

3. Show that S_k is generated by $s_1, \ldots, s_{k-1}$ with relations
$$s_i^2 = 1, \qquad s_is_{i+1}s_i = s_{i+1}s_is_{i+1}, \qquad \text{and} \qquad s_is_j = s_js_i \ \text{ for } \ j \neq i, i \pm 1.$$

4. In the group algebra of the symmetric group $\mathbb{C}S_k$ define
$$m_j = s_{1j} + s_{2j} + \cdots + s_{j-1,j},$$
where s_{ij} is the transposition that switches i and j. Let $m_1 = 0$.

    a. Show that $m_1 + \cdots + m_k$ is an element of the center of $\mathbb{C}S_k$.

    b. Show that $m_im_j = m_jm_i$ for all $1 \leq i, j \leq k$.

5. Construct explicitly some modules for $\mathbb{C}S_k$. which have a basis of eigenvectors for the m_i. Do this by describing, explicitly, the action of the s_i and the m_i on the basis vectors.

    a. Be sure to prove that the modules you construct are S_k-modules (by showing that the formulas for the action satisfy the necessary relations).

    b. Show that the modules you have constructed are irreducible.

    c. Show that the modules you constructed are pairwise nonisomorphic.

    d. Show that you have constructed all the irreducible S_k-modules.

6. Use the modules constructed in Problem 5 (or find an alternative method) to determine (with proof) the Bratelli diagram for the tower of algebras

$$\mathbb{C}S_1 \subseteq \mathbb{C}S_2 \subseteq \mathbb{C}S_3 \subseteq \cdots .$$