Representation Theory Lecture 11 19.10.2008 ①

A morphism $f: X \to Y$ of spaces provides

$$df: T_x(X) \longrightarrow T_{f(x)}(Y), \quad \text{for } x \in X.$$

Let G be a Lie group or algebraic group.

The conjugation action of G on G is given by

$$\begin{aligned} Inn_g: G &\longrightarrow G \\ h &\longmapsto ghg^{-1} \end{aligned} \quad \text{for } g \in G.$$

The differential of these maps gives the

Adjoint action of G on $\mathfrak{g} = T_1(G)$

$$Ad_g: \mathfrak{g} \to \mathfrak{g}, \quad \text{for } g \in G.$$

Example

$G = GL_n$ has Lie algebra $\mathfrak{gl}_n = M_n(\mathbb{C})$

and the exponential map is

$$\begin{aligned} \mathfrak{gl}_n &\longrightarrow GL_n \\ x &\longmapsto e^x \end{aligned} \quad \text{where}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

SO_n, O_n, Sp_n etc. are subgroups of GL_n and $\mathfrak{so}_n, \mathfrak{o}_n, \mathfrak{sp}_n$ etc are Lie subalgebras of $\mathfrak{gl}_n$.

Since

$$\begin{aligned} Inn_g : GL_n &\to GL_n \\ h &\mapsto ghg^{-1} \\ e^{ty} &\mapsto ge^{ty}g^{-1} \end{aligned}$$

and

$$ge^{tx}g^{-1} = g\left(1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \cdots\right)g^{-1} = e^{t(gxg^{-1})}$$

it follows that

$$\begin{aligned} Ad_g : \mathfrak{gl}_n &\to \mathfrak{gl}_n \\ x &\mapsto gxg^{-1}. \end{aligned}$$

Let M be a G-module,

$$\begin{aligned} \rho : G &\to GL(M) \\ g &\mapsto \rho(g) \\ e^{tx} &\mapsto \rho(e^{tx}), \end{aligned}$$

the corresponding representation of G. If

$$\rho(x) = \frac{d}{dt}\rho(e^{tx})\Big|_{t=0} \quad \text{then } \rho(e^{tx}) = e^{t\rho(x)}$$

and we get a representation of $\mathfrak{g}$ on M

$$\begin{aligned} \rho : \mathfrak{g} &\to End(M) \\ x &\mapsto \rho(x). \end{aligned}$$

The group G acts on $\mathfrak{g}$ by the Adjoint action

~~and~~
$$\begin{aligned} Ad_g : \mathfrak{g} &\longrightarrow \mathfrak{g} \\ x &\longmapsto gxg^{-1} \end{aligned}$$
and

the Lie algebra $\mathfrak{g}$ acts on $\mathfrak{g}$ by the <u>adjoint</u> <u>action</u>

$$\begin{aligned} ad_y : \mathfrak{g} &\longrightarrow \mathfrak{g} \\ x &\longmapsto [y,x], \quad \text{for } y \in \mathfrak{g} \end{aligned}$$

since

$$Ad_{e^{ty}}(x) = e^{ty} x e^{-ty} = \left(1 + ty + \frac{t^2 y^2}{2!} + \cdots\right) x \left(1 - ty + \frac{t^2 y^2}{2!} - \frac{t^3 y^3}{3!} + \cdots\right)$$

$$= x + t(yx - xy) + \frac{t^2}{2!}(y^2 x - 2yxy + xy^2) + \cdots$$

$$= \left(e^{t\, ad_y}\right)(x).$$

Note: $(ad_y)^2(x) = [y,[y,x]] = [y,(yx-xy)] = y^2x - yxy - yxy + xy^2$.

So we have three actions:

conjugation action
$$\begin{aligned} Inn_g : G &\longrightarrow G \\ h &\longmapsto ghg^{-1}. \end{aligned}$$

Adjoint action
$$\begin{aligned} Ad_g : \mathfrak{g} &\to \mathfrak{g} \\ x &\mapsto gxg^{-1} \end{aligned}$$

adjoint action
$$\begin{aligned} ad_y : \mathfrak{g} &\to \mathfrak{g} \\ x &\longmapsto [y,x]. \end{aligned}$$

Let M be a G-module. The dual vector space

$$M^* = \mathrm{Hom}(M, \mathbb{C}) = \{\varphi : M \to \mathbb{C} \mid \varphi \text{ is linear}\}$$

is a G-module with action given by

$$(g\varphi)(m) = \varphi(g^{-1}m), \quad \text{for } g \in G,\ m \in M.$$

Since

$$(e^{tx}\varphi)(m) = \varphi(e^{-tx}m),$$

if M is a $\mathfrak{g}$-module, then M^* is a $\mathfrak{g}$-module with action given by

$$(x\varphi)(m) = \varphi(-xm), \quad \text{for } x \in \mathfrak{g},\ m \in M.$$

Thus we have 5-actions:

conjugation $\quad \mathrm{Int}_g : G \to G$, $\quad h \mapsto ghg^{-1}$

Adjoint $\quad \mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$, $\quad x \mapsto gxg^{-1}$ $\qquad$ coAdjoint $\quad \mathrm{Ad}_g^* : \mathfrak{g}^* \to \mathfrak{g}^*$

adjoint $\quad \mathrm{ad}_y : \mathfrak{g} \to \mathfrak{g}$, $\quad x \mapsto [y, x]$ $\qquad$ coadjoint $\quad \mathrm{ad}_y^* : \mathfrak{g}^* \to \mathfrak{g}^*$

Tori and Cartan subalgebras

Let G be an algebraic group.

A torus H is a subgroup of G such that

$$H \simeq \underbrace{\mathbb{C}^\times \times \cdots \times \mathbb{C}^\times}_{n}, \quad \text{for some } n \in \mathbb{Z}_{>0}.$$

Let K be a Lie group.

A torus T is a subgroup of K such that

$$K \simeq \underbrace{S^1 \times \cdots \times S^1}_{n}, \quad \text{for some } n \in \mathbb{Z}_{>0}$$

where $S^1 = U(1) = \{ z \in \mathbb{C}^\times \mid z\bar{z} = 1 \}$

Let $\mathfrak{g}$ be a Lie algebra.

An abelian Lie subalgebra is a Lie subalgebra $\mathfrak{h}$ such that

$$[h_1, h_2] = 0, \quad \text{for } h_1, h_2 \in \mathfrak{h}.$$

A Cartan subalgebra is a maximal abelian Lie subalgebra of $\mathfrak{g}$.

Example A maximal torus of GL_n is

$$H = \left\{ \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix} \;\middle|\; x_1, \ldots, x_n \in \mathbb{C}^\times \right\}.$$

A Cartan subalgebra of $\mathfrak{gl}_n$ is

$$\mathfrak{h} = \left\{ \begin{pmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_n \end{pmatrix} \;\middle|\; h_1, \ldots, h_n \in \mathbb{C} \right\}.$$

Note that $\mathfrak{h} = Lie(H) = T_1(H)$.

Since $\mathfrak{h} \subseteq \mathfrak{g}$ and $H \subseteq G$,

H acts on G by conjugation

H acts on $\mathfrak{g}$ by the Adjoint action

$\mathfrak{h}$ acts on $\mathfrak{g}$ by the adjoint action.

The irreducible (rational) representations of H are

$$X^\mu = X^{\mu_1\varepsilon_1 + \cdots + \mu_n\varepsilon_n} = X^{\mu_1\varepsilon_1} \cdots X^{\mu_n\varepsilon_n}$$

$$= (X^{\varepsilon_1})^{\mu_1} \cdots (X^{\varepsilon_n})^{\mu_n}, \quad \text{with } \mu_1, \ldots, \mu_n \in \mathbb{Z}$$

where

$$X^{\varepsilon_i} : H \longrightarrow \mathbb{C}^\times$$

$$\begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix} \longmapsto x_i.$$

The irreducible representations of $\mathfrak{h}$ are

$$\mu : \mathfrak{h} \to \mathbb{C}, \quad \text{so that} \quad \mu \in \mathfrak{h}^*,$$

and

$$\mu = \mu_1 \varepsilon_1 + \cdots + \mu_n \varepsilon_n, \text{ with } \mu_1, \ldots, \mu_n \in \mathbb{C} \text{ and}$$

$$\begin{aligned} \varepsilon_i : \mathfrak{h} &\longrightarrow \mathbb{C} \\ \begin{pmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_n \end{pmatrix} &\longmapsto h_i . \end{aligned}$$

Hence

$\mathfrak{h}^*$ indexes irreducible reps of $\mathfrak{h}$, and

$\{X^\mu \mid \mu \in \mathfrak{h}^*_{\mathbb{Z}}\}$ are the irreducible reps of H.

Weights and roots

Let M be a G-module and

$X^\mu : H \to \mathbb{C}^*$ and irreducible representation of H.

The μ-weight space of M is

$$M_\mu = \left\{ m \in M \,\middle|\, \begin{array}{l} \text{for each } t \in H, \\ tm = X^\mu(t) m \end{array} \right\}$$

$$= \{ m \in M \mid \text{for each } h \in \mathfrak{h},\ hm = \mu(h) m \}$$

The <u>generalized</u> <u>μ-weight space</u> of M is

$$M_\mu^{gen} = \left\{ m \in M \;\middle|\; \begin{array}{l} \text{for each } t \in H, \\ (t - X^M(t))^\ell m = 0, \text{ for some } \ell \in \mathbb{Z}_{>0} \end{array} \right\}$$

$$= \left\{ m \in M \;\middle|\; \begin{array}{l} \text{for each } h \in \mathfrak{h} \\ (h - \mu(h))^\ell m = 0, \text{ for some } \ell \in \mathbb{Z}_{>0} \end{array} \right\}.$$

Note that $M_\mu \subseteq M_\mu^{gen}$ and $M_\mu^{gen} \neq 0$ implies $M_\mu \neq 0$.

$$M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu^{gen}$$

The <u>weights</u> of M are the μ such that $M_\mu \neq 0$.

The adjoint representation $\mathfrak{g}$ (G acts on $\mathfrak{g}$ or $\mathfrak{g}$ acts on $\mathfrak{g}$) is a G-module.

The <u>roots</u> of G (or $\mathfrak{g}$) are the nonzero weights of $\mathfrak{g}$.

Note that $\mathfrak{g}_0 = \mathfrak{h}$, so the "interesting" weights of $\mathfrak{g}$ are the nonzero ones.

Example

$\mathfrak{g} = \mathfrak{gl}_n$ has basis $\{E_{ij} \mid 1 \le i,j \le n\}$

If

$$t = \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix} \in H \quad \text{and} \quad h = \begin{pmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_n \end{pmatrix} \in \mathfrak{h}.$$

then

$$t E_{ij} t^{-1} = x_i x_j^{-1} E_{ij} = X^{\varepsilon_i - \varepsilon_j}(t) E_{ij} . \quad \text{and}$$

$$[h, E_{ij}] = (h_i - h_j) E_{ij} = (\varepsilon_i - \varepsilon_j)(h) E_{ij} .$$

and, hence

$$\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{C} E_{ij}, \quad \text{for } 1 \le i,j \le n \quad (\text{and } \mathfrak{g}_0 = \mathfrak{h}).$$

Note that $\mathfrak{g}$ contains lots of $\mathfrak{sl}_2$-subalgebras

$$E_{ij} = \begin{array}{c|c} & j \\ \hline i & \begin{pmatrix} 0 & & 1 \\ & \ddots & \\ & & 0 \end{pmatrix} \end{array} \qquad E_{ji} = \begin{array}{c|c} & i \\ \hline j & \begin{pmatrix} 0 & & \\ & \ddots & \\ 1 & & 0 \end{pmatrix} \end{array} \qquad h_{ij} = \begin{pmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & 1 & & & \\ & & & & 0 \cdots 0 & & \\ & & & & & -1 & \\ & & & & & & 0 \cdots 0 \end{pmatrix}.$$

A <u>one parameter subgroup</u> is an "imbedding" of $\mathbb{C}^\times$ in G

An <u>"SL_2 imbedding"</u> is a homomorphism

$$\varphi_\alpha : SL_2(\mathbb{C}) \longrightarrow G.$$

The Weyl group is

$$W_0 = \frac{N(H)}{H}$$ where $N(H)$ is the normalizer of H in G,

$$N(H) = \{ n \in G \mid nHn^{-1} = H \}.$$

The Weyl group acts on H, by conjugation, and W_0 acts $\mathfrak{h}$ and, hence

W_0 acts on $\mathfrak{h}^*$, and

$$w : M_\mu \longrightarrow M_{w\mu}, \quad \text{for } w \in W.$$