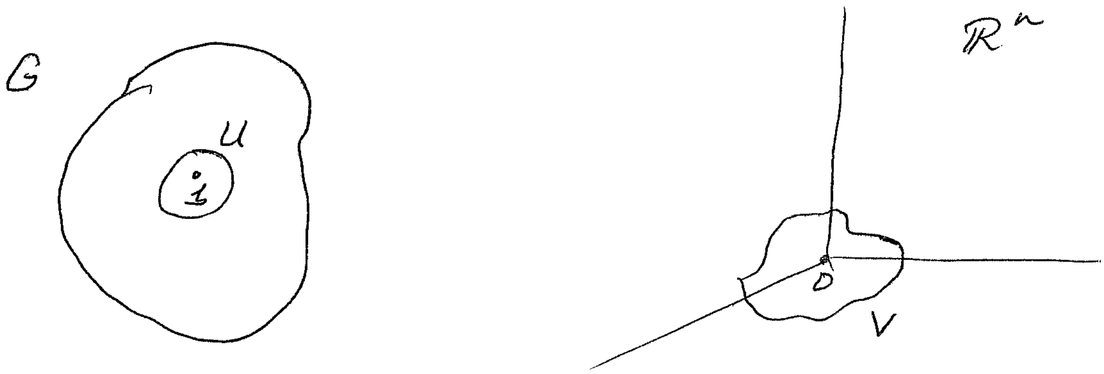
Representation Theory Lecture 10 — 14.10.2008

Lie algebras

A *Lie group* is a group that is also a manifold i.e. a topological group that is locally isomorphic to $\mathbb{R}^n$

$$\varphi : U \xrightarrow{\sim} V$$



U is an open neighborhood of 1 in G

V is an open neighborhood of 0 in $\mathbb{R}^n$.

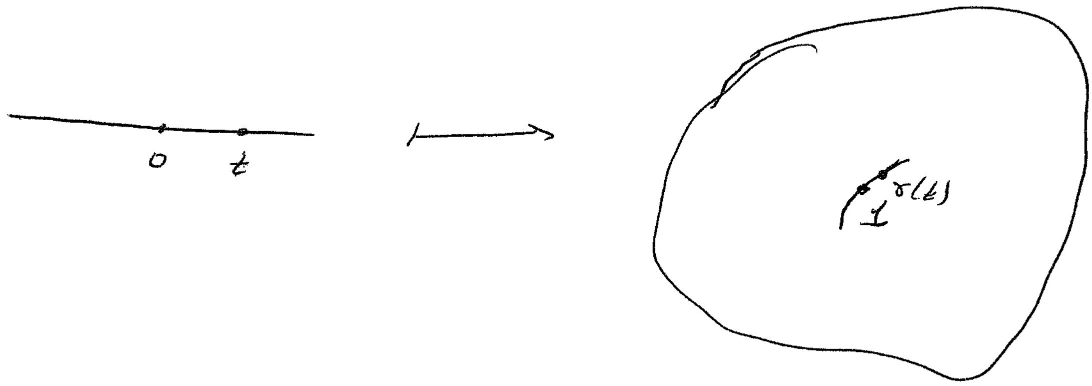
If G is connected then G is generated by the elements of U.

The *exponential map* is a smooth homomorphism

$$\begin{aligned} \mathfrak{g} &\longrightarrow G \\ 0 &\longmapsto 1 \end{aligned}$$

which is a homeomorphism on a neighborhood of 0. The Lie algebra $\mathfrak{g}$ contains the structure of G in a neighborhood of the identity.

A <u>one parameter subgroup</u> of G is a smooth group homomorphism $\gamma : \mathbb{R} \to G$.



<u>Examples:</u>

(1) $$\gamma : \mathbb{R} \longrightarrow GL_n(\mathbb{R})$$
$$t \longmapsto 1 + tE_{ij} = x_{ij}(t), \quad \text{for } i \neq j.$$

Note that
$$x_{ij}(t)\, x_{ij}(s) = x_{ij}(s+t)$$

since $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & t+s \\ 0 & 1 \end{pmatrix}$.

(2) $$\gamma : \mathbb{R} \longrightarrow GL_n(\mathbb{R})$$
$$t \longmapsto \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & e^t & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} = h_i(e^t).$$

Note that
$$h_i(e^t)\, h_i(e^s) = h_i(e^{t+s}).$$

Let G be a Lie group. The <u>ring of functions</u> on G is

$$C^\infty(G) = \{f: G \to \mathbb{R} \mid f \text{ is smooth at } g \text{ for all } g \in G\}$$

where

f is <u>smooth at g</u> if $\left.\dfrac{d^k f}{dx^k}\right|_{x=g}$ exists for all $k \in \mathbb{Z}_{>0}$.

Let $g \in G$. A <u>tangent vector</u> to G at g is a linear map $\eta: C^\infty(G) \to \mathbb{R}$ such that

$$\eta(f_1 f_2) = f_1(g)\,\eta(f_2) + \eta(f_1)\,f_2(g),$$

for all $f_1, f_2 \in C^\infty(G)$. A <u>vector field</u> is a linear map $\partial: C^\infty(G) \to C^\infty(G)$ such that

$$\partial(f_1 f_2) = f_1\,\partial(f_2) + \partial(f_1)\,f_2,$$

for $f_1, f_2 \in C^\infty(G)$. A <u>left invariant vector field</u> on G is a vector field $\partial: C^\infty(G) \to C^\infty(G)$ such that

$$L_g \xi = \xi L_g, \quad \text{for all } g \in G,$$

where $L_g: C^\infty(G) \to C^\infty(G)$ is given by

$$(L_g f)(x) = f(g^{-1}x), \quad \text{for } f \in C^\infty(G),\ g, x \in G.$$

The Lie algebra of G is the vector space $\mathfrak{g}$ of left invariant vector fields on G with bracket

$$[\partial_1, \partial_2] = \partial_1\partial_2 - \partial_2\partial_1 .$$

A one-parameter subgroup of G is a smooth group homomorphism $\gamma: \mathbb{R} \to G$. If γ is a one-parameter subgroup of G define

$$\frac{d\, f(\gamma(t))}{dt} = \lim_{h\to 0} \frac{f(\gamma(t+h)) - f(\gamma(t))}{h}$$

and let γ_1 be the tangent vector at 1 given by

$$\gamma_1(f) = \left.\frac{df(\gamma(t))}{dt}\right|_{t=0}.$$

Identify the vector spaces

$\{$left invariant vector fields on $G\}$,

$\{$one parameter subgroups of $G\}$, and

$\{$tangent vectors at $1\}$,

by the vector space isomorphisms

$$\left\{\begin{matrix}\text{one parameter}\\ \text{subgroups}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{tangent vectors}\\ \text{at } 1\end{matrix}\right\}$$

$$\gamma \longmapsto \gamma_1$$

and

$$\left\{\begin{matrix}\text{left invariant}\\ \text{vector fields}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{tangent vectors}\\ \text{at } 1\end{matrix}\right\}.$$

$$\xi \longmapsto \xi_1$$

where

$$\xi_1(f) = (\xi f)(1).$$

The <u>exponential map</u> is

$$\begin{matrix} \mathfrak{g} \longrightarrow G \\ tX \longmapsto e^{tX} \end{matrix} \quad \text{where } e^{tX} = \gamma(t),$$

where γ is the 1-parameter subgroup corresponding to X.

<u>Examples:</u> The Lie algebra $\mathfrak{gl}_n$ is

$$\mathfrak{gl}_n = \{x \in M_n(\mathbb{C})\} \text{ with bracket}$$

$$[x_1, x_2] = x_1 x_2 - x_2 x_1.$$

Our favorite basis of $\mathfrak{gl}_n$ is

$$\{E_{ij} \mid 1 \le i,j \le n\}.$$

The exponential map is

$$\begin{aligned} \mathfrak{gl}_n &\longrightarrow GL_n \\ tX &\longmapsto e^{tX}, \end{aligned} \qquad \text{where}$$

$$e^A = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

for a matrix A. In fact

$$e^{tE_{ij}} = 1 + tE_{ij} = \begin{array}{c c} & \begin{array}{c} \quad\quad j \end{array} \\ i & \begin{pmatrix} 1 & & t \\ & \ddots & \\ & & 1 \end{pmatrix} \end{array} \quad \text{for } i \neq j$$

and

$$e^{tE_{ii}} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & e^t & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} = h_i(e^t).$$

(b) If $n=1$ the exponential map

$$\begin{aligned} \mathbb{C} &\longrightarrow \mathbb{C}^\times \\ tx &\longmapsto e^{tx} \end{aligned} \quad \text{is a homeomorphism}$$

from a neighborhood of 0 to a neighborhood of 1.

In fact, if $e(t) = a_0 + a_1 t + a_2 t^2 + \cdots$ and

$$e(s+t) = e(s)e(t),$$

then

$$e(s+t) = a_0 + a_1(s+t) + a_2(s+t)^2 + a_3(s+t)^3 + \dots$$

$$\begin{aligned} = \; & a_0 \\ & a_1 s + a_1 t \\ & a_2 s^2 + 2a_2 s t + a_2 t^2 \\ & a_3 s^3 + 3a_3 s^2 t + 3a_3 s t^2 + a_3 t^3 \\ & \vdots \end{aligned}$$

and

$$e(s)e(t) = (a_0 + a_1 s + a_2 s^2 + a_3 s^3 + \dots)(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots)$$

$$\begin{aligned} = \; & a_0^2 \\ & a_0 a_1 s + a_0 a_1 t \\ & a_0 a_2 s^2 + a_1^2 s t + a_0 a_2 t^2 \\ & a_0 a_3 s^3 + a_2 a_1 s^2 t + a_1 a_2 s t^2 + a_0 a_3 t^3 \\ & \vdots \end{aligned}$$

Hence $e(s+t) = e(s)e(t)$ only if

$$a_0 a_1 = a_1, \quad 2a_2 = a_1^2, \quad 3a_3 = a_1 a_2, \quad 4a_3 = a_1 a_3, \dots$$

so that

$$a_0 = 1, \quad a_2 = \frac{a_1^2}{2}, \quad a_3 = \frac{a_1^3}{3!}, \quad a_4 = \frac{a_1^4}{4!}, \dots$$

and

$$e(t) = 1 + a_1 t + \frac{a_1^2}{2} t^2 + \frac{a_1^3 t^3}{3!} + \dots = e^{a_1 t}.$$

So $\begin{aligned} \mathbb{C} &\to \mathbb{C}^* \\ z &\mapsto e^z \end{aligned}$ is the "unique" smooth homomorphism $\mathbb{C} \to \mathbb{C}^*$

The Lie groups SL_2 and SU_2

$$SL_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; ad - bc = 1 \right\}$$

One parameter subgroups are

$$x_{12}(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad x_{21}(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$h_{\alpha_1}(e^t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

and the Lie algebra

$$sl_2 = \{ x \in gl_2 \mid \mathrm{tr}\, x = 0 \}$$

has basis $\{x, y, h\}$ where

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Lie algebra sl_2 is presented by generators x, y, h and relations

$$[x,y] = h, \quad [h,x] = 2x \text{ and } [h,y] = -2y.$$

The group SL_2 is presented by generators

$$x_\alpha(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad x_{-\alpha}(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \qquad t \in \mathbb{F}$$

with relations

$$x_\alpha(s+t) = x_\alpha(s)\, x_\alpha(t), \qquad h_\alpha(c_1 c_2) = h_\alpha(c_1)\, h_\alpha(c_2)$$

and

$$n_\alpha(t)\, x_\alpha(u)\, n_\alpha(-t) = x_{-\alpha}(-t^{-2} u)$$

where

$$n_\alpha(t) = x_\alpha(t)\, x_{-\alpha}(-t^{-1})\, x_\alpha(t) \quad \text{and}$$

$$h_\alpha(t) = n_\alpha(t)\, n_\alpha(-1)$$

Note that

$$n_\alpha(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & t \\ -t^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix} \quad \text{and}$$

$$h_\alpha(t) = \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

The maximal compact subgroup of $SL_2(\mathbb{C})$ is

$$SU_2 = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{matrix} g\bar{g}^t = 1 \\ \det(g) = 1 \end{matrix} \right\}$$

$$= \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \;\middle|\; a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1 \right\}$$

since

$$g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{and} \quad \bar{g}^t = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}.$$

The Lie algebra su_2 is

$$su_2 = \{x \in gl_2(\mathbb{C}) \mid \mathrm{tr}\, x = 0 \text{ and } x + \bar{x}^t = 0\}$$
$$= \mathbb{R}\text{-span}\{i\sigma^x, i\sigma^y, i\sigma^z\}, \text{ where}$$

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the <u>Pauli</u> <u>matrices</u>
and

$$[\sigma^x, \sigma^y] = 2i\sigma^z, \quad [\sigma^y, \sigma^z] = 2i\sigma^x, \quad [\sigma^z, \sigma^x] = 2i\sigma^y.$$

Then $sl_2(\mathbb{C})$ is the complexification of su_2,

$$sl_2(\mathbb{C}) = \mathbb{C} \otimes_{\mathbb{R}} su_2$$

and the change of basis is given by

$$\sigma^x = x + y, \quad \sigma^y = -ix + iy, \quad \sigma^z = h$$

$$x = \tfrac{1}{2}(\sigma^x + i\sigma^y), \quad y = \tfrac{1}{2}(\sigma^x - i\sigma^y)$$

Note that

$GL_1(\mathbb{C}) = \mathbb{C}^\times$ has maximal compact subgroup

$U(1) = S^1 = \{z \in \mathbb{C}^\times \mid z\bar{z} = 1\}$

and $SL_1(\mathbb{C}) = \{\pm 1\}$ has maximal compact subgroup

$SU(1) = \{1\}$.