Representation Theory Lecture 9 7 October 2008

Spaces

A Lie group is a group G that is also a manifold such that the maps

$$\begin{array}{ccc} G\times G \to G & \text{and} & G \to G \\ (g_1,g_2)\mapsto g_1g_2 & & g \mapsto g^{-1} \end{array}$$

are morphisms of manifolds.

An algebraic group is a group G that is also a variety such that the maps

$$\begin{array}{ccc} G\times G \to G & \text{and} & G \to G \\ (g_1,g_2)\mapsto g_1g_2 & & g \mapsto g^{-1} \end{array}$$

are morphisms of varieties.

A topological group is a group G that is also a topological space such that

$$\begin{array}{ccc} G\times G \to G & \text{and} & G \to G \\ (g_1,g_2)\mapsto g_1g_2 & & g \to g^{-1} \end{array}$$

are morphisms of topological spaces.

A group scheme is a group G that is also a scheme such that

$$\begin{array}{ccc} G\times G \to G & \text{and} & G \to G \\ (g_1,g_2)\mapsto g_1g_2 & & g \mapsto g^{-1} \end{array}$$

are morphisms of schemes.

A <u>complex Lie group</u> is a group G that is also a complex manifold such that

$$\begin{array}{ccc} G\times G \longrightarrow G & \text{and} & G \longrightarrow G \\ (g_1,g_2) \longmapsto g_1 g_2 & & g \longmapsto g^{-1} \end{array}$$

are morphisms of complex manifolds

<u>Remarks:</u> (a) morphisms of manifolds are called <u>smooth functions</u>. Lie groups have $\mathbb{R}$ in them in a crucial way.

(b) morphisms of varieties are called <u>regular functions</u>. Algebraic groups usually need to be based on an <u>algebraically closed field</u>. A <u>variety</u> is a topological space with is locally isomorphic to an <u>affine variety</u>.

(c) morphisms of topological spaces are called continous functions.

(d) schemes are varieties over $\mathbb{Z}$.

(e) Complex manifolds are not manifolds.

(f) morphisms of Lie groups, morphisms of algebraic groups, morphisms of topological groups, morphisms of schemes are all different things

There are equivalences of categories:

$$
\begin{tikzcd}
\left\{ \begin{array}{c} \text{connected reductive} \\ \text{complex algebraic groups} \end{array} \right\} \arrow[d, leftrightarrow] & G \arrow[d, mapsto] \\
\left\{ \begin{array}{c} \text{connected compact} \\ \text{Lie groups} \end{array} \right\} \arrow[d, leftrightarrow] & K \arrow[d, mapsto] \\
\left\{ \begin{array}{c} \text{complex semisimple} \\ \text{Lie algebras} \end{array} \right\} \arrow[d, leftrightarrow] & \mathfrak{g} \arrow[d, mapsto] \\
\{\mathbb{Z}\text{-reflection groups}\} \arrow[d, leftrightarrow] & (W_0, \mathfrak{h}^*_{\mathbb{Z}}) \\
\{\text{Dynkin diagrams}\} &
\end{tikzcd}
$$

Examples: GL_n, SL_n, PGL_n.

$$GL_n(\mathbb{C}) = \{ g \in M_n(\mathbb{C}) \mid g \text{ is invertible} \}.$$

Let V be a vector space over $\mathbb{F}$.

$$GL(V) = \{ g \in End(V) \mid g \text{ is invertible} \}.$$

$GL_n(\mathbb{C})$ is a complex algebraic group.
$GL_n(\mathbb{F})$ is an algebraic group
GL_n is ???

The group homomorphism

$$\det : GL_n(\mathbb{F}) \longrightarrow \mathbb{F}^\times$$

is a 1-dimensional representation (character) of $GL_n(\mathbb{F})$.

$$SL_n(\mathbb{F}) = \ker(\det)$$
$$= \{ g \in GL_n(\mathbb{F}) \mid \det(g) = 1 \}$$

The <u>center</u> of $GL_n(\mathbb{F})$ is

$$Z(GL_n) = \{ c \cdot Id \mid c \in \mathbb{F}^\times \}$$

$$PGL_n = \frac{GL_n(\mathbb{F})}{Z(GL_n(\mathbb{F}))}$$

$GL_n(\mathbb{C})$ is a complex reductive alg. gp

$SL_n(\mathbb{C})$ is a complex semisimple alg. gp.

$PGL_n(\mathbb{C})$ is a complex semisimple alg. gp.

In spite of

$$SL_n(\mathbb{C}) \subseteq GL_n(\mathbb{C}) \quad \text{and} \quad GL_n(\mathbb{C}) = SL_n(\mathbb{C}) \cdot \mathbb{C}^*$$

and

$$1 \to SL_n(\mathbb{C}) \to GL_n(\mathbb{C}) \xrightarrow{\det} \mathbb{C}^* \to 1$$

being exact,

$$PGL_n(\mathbb{C}) \not\cong SL_n(\mathbb{C}).$$

$$Z(SL_n(\mathbb{C})) = \{n^{\text{th}} \text{ roots of } 1\} = \mu_n(\mathbb{C}).$$

Examples U_n, O_n, Sp_n

The unitary group

$$U(n) = \{ g \in GL_n(\mathbb{C}) \mid g\bar{g}^t = 1 \}$$

where $\bar{g} = (\bar{g}_{ij})$ if $g = (g_{ij})$.

The orthogonal group

$$O_n(\mathbb{C}) = \{ g \in GL_n(\mathbb{C}) \mid g g^t = 1 \}$$

The symplectic group

$$Sp_{2n}(\mathbb{C}) = \{ g \in GL_n(\mathbb{C}) \mid g J g^t = J \}.$$

where

$$J = \left(\begin{array}{c|c} 0 & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \\ \hline \begin{matrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{matrix} & 0 \end{array} \right) \quad \text{or} \quad J = \left(\begin{array}{c|c} 0 & \begin{matrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{matrix} \\ \hline \begin{matrix} 0 & & -1 \\ & \iddots & \\ -1 & & 0 \end{matrix} & 0 \end{array} \right)$$

Let V be a vector space over $\mathbb{F}$.

A symmetric bilinear form on V is a map

$$\langle \, , \rangle : V \times V \longrightarrow \mathbb{F}$$
$$(v_1, v_2) \longmapsto \langle v_1, v_2 \rangle \quad \text{such that}$$

(a) $\langle , \rangle$ is bilinear, i.e.

$$\langle v_1+v_2, v_3\rangle = \langle v_1, v_3\rangle + \langle v_2, v_3\rangle,$$

$$\langle v_1, v_2+v_3\rangle = \langle v_1, v_2\rangle + \langle v_1, v_3\rangle,$$

$$\langle cv_1, v_2\rangle = c\langle v_1, v_2\rangle, \quad \text{and}$$

$$\langle v_1, cv_2\rangle = c\langle v_1, v_2\rangle,$$

for $v_1, v_2, v_3 \in V$, $c \in \mathbb{F}$,

(b) $\langle\,\rangle$ is symmetric, i.e.

$$\langle v_1, v_2\rangle = \langle v_2, v_1\rangle, \quad \text{for } v_1, v_2 \in V.$$

The <u>orthogonal group</u> is

$$O_n(\mathbb{F}) = O(V, \langle\,\rangle) = O(\langle , \rangle)$$

$$= \left\{ g \in GL(V) \;\middle|\; \begin{matrix} \langle gv_1, gv_2\rangle = \langle v_1, v_2\rangle \\ \text{for } v_1, v_2 \in V \end{matrix} \right\},$$

the group of invertible linear transformations "preserving the metric".

A <u>skew symmetric</u> form on V is a map

$$\langle , \rangle : V \times V \longrightarrow \mathbb{F} \quad \text{such that}$$

(a) $\langle , \rangle$ is bilinear,

(b) $\langle v_2, v_1\rangle = -\langle v_1, v_2\rangle$, for $v_1, v_2 \in V$.

The *symplectic group* is

$$Sp_n(\mathbb{F}) = Sp(V) = Sp(V, \langle\ \rangle) = Sp(\langle\ \rangle)$$

$$= \left\{ g \in GL(V) \;\middle|\; \begin{matrix} \langle gv_1, gv_2 \rangle = \langle v_1, v_2 \rangle \\ \text{for } v_1, v_2 \in V \end{matrix} \right\}$$

Let $\bar{\ } : \mathbb{F} \to \mathbb{F}$ be an involution.
$z \mapsto \bar{z}$

A *sesquilinear form*, or *Hermitian form*, is a map
$\langle\ ,\ \rangle : V \times V \to \mathbb{F}$ such that

(a) $\langle\ \rangle$ is not bilinear, instead

$\langle v_1 + v_2, v_3 \rangle = \langle v_1, v_3 \rangle + \langle v_2, v_3 \rangle$,

$\langle v_1, v_2 + v_3 \rangle = \langle v_1, v_2 \rangle + \langle v_1, v_3 \rangle$,

$\langle c v_1, v_2 \rangle = c \langle v_1, v_2 \rangle$, and

$\langle v_1, c v_2 \rangle = \bar{c} \langle v_1, v_2 \rangle$,

for $v_1, v_2, v_3 \in V$ and $c \in \mathbb{F}$

(b) $\langle v_2, v_1 \rangle = \overline{\langle v_1, v_2 \rangle}$, for $v_1, v_2 \in V$.

The *unitary group*

$$U_n = \left\{ g \in GL(V) \;\middle|\; \begin{matrix} \langle v_1, v_2 \rangle = \langle gv_1, gv_2 \rangle \\ \text{for all } v_1, v_2 \in V \end{matrix} \right\}.$$

Maximal compacts and maximal tori

$$\left\{\begin{array}{l}\text{connected reductive}\\ \text{linear algebraic}\\ \text{groups over } \mathbb{C}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{compact}\\ \text{connected Lie}\\ \text{groups}\end{array}\right\}$$

$$G \longmapsto K,$$

where K is the maximal compact subgroup of G

A torus in a compact Lie group is a subgroup isomorphic to $S^1 \times \cdots \times S^1$.

A torus in an algebraic group is a subgroup isomorphic to $\mathbb{F}^\times \times \cdots \times \mathbb{F}^\times$.

$GL_1(\mathbb{F}) = \mathbb{F}^\times$ and $GL_1(\mathbb{C}) = \mathbb{C}^\times$ has maximal compact subgroup

$$U(1) = \{ z \in \mathbb{C}^\times \mid z\bar{z} = 1 \} = S^1.$$

So the maximal compact subgroup of $\mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$ is $S^1 \times \cdots \times S^1$.

$$\xrightarrow{\text{maximal compact}}$$

$$\begin{array}{c} \Big\downarrow \text{maximal torus} \end{array} \qquad \begin{tikzcd} G \arrow[r, mapsto] \arrow[d, mapsto] & K \arrow[d, mapsto] \\ T \arrow[r, mapsto] & T_K \end{tikzcd}$$

SU(2)

$$SU(2) = \{ g \in SL_2(\mathbb{C}) \mid g\bar{g}^t = 1 \}$$

If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$ then

$$g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \bar{g}^t = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} \quad \text{so that}$$

$$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \quad \text{with} \quad |a|^2 + |b|^2 = 1.$$

So

$$SU(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \;\middle|\; a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1 \right\}.$$

Define an involution

$$\begin{aligned} \sigma : SL_2(\mathbb{C}) &\longrightarrow SL_2(\mathbb{C}) \\ g &\longmapsto (\bar{g}^t)^{-1} \end{aligned}$$

Then

$$SU(2) = SL_2(\mathbb{C})^\sigma = \{ g \in SL_2(\mathbb{C}) \mid \sigma(g) = g \}$$

Let G be a complex reductive algebraic group.

$$\begin{aligned} \sigma : G &\longrightarrow G \\ x_\alpha(t) &\longmapsto x_{-\alpha}(-\bar{t}) \\ h_\alpha(t) &\longmapsto h_\alpha(\bar{t}^{-1}) \end{aligned} \quad \text{an involution}$$

Then

$K = G^\sigma = \{ g \in G \mid \sigma(g) = g \}$ is a maximal compact subgroup.