REPRESENTATION THEORY

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Abstract. Notes from Arun Ram's 2008 course at the University of Melbourne.

8. Week 7

What have we done so far in this course?

(1) Representation Theory = study of A-modules; A is an algebra (first a vector space). ie, representation theory is advanced linear algebra.
(2) Braid-like examples: Temperley-Lieb, Hecke, symmetric group, etcetera. Good control of simple modules from Bratteli diagram techniques.
(3) $\mathcal{U}(\mathfrak{sl}_2)$ and $\mathcal{U}_q(\mathfrak{sl}_2)$ are infinite dimensional algebras. Main tool is tensor products. Magic: $\mathfrak{sl}_2$ tensor product produces the Bratteli diagram for TL.
(4) Crystals: not algebras or vector spaces, just sets with operators. They do have a tensor product operation. Magic: Bratteli diagram for TL appears again!

Reason: There is an equivalnce of tensor categories

$$\{\mathfrak{sl}_2 \text{ crystals}\} \longleftrightarrow \{\text{fin dim } \mathcal{U}_q(\mathfrak{sl}_2) \text{ modules}\}$$

This type of equivalence is a feature of "semisimple Lie theory" (Lie groups, Lie algebras, algebraic groups, quantum groups).

Next six weeks: Lie Theory. The main theorem is the Weyl character formula and what makes it work.

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Theorem 8.1 (Amazing Theorem)**.** *There is an equivalence of categories*

$$\{\textit{connected compact Lie groups}\} \longleftrightarrow \{\mathbb{Z}\textit{-reflection groups}\}$$

Example. Connected compact Lie groups: $\mathbb{R}$, $GL_n(\mathbb{C})$, $SO_n(\mathbb{C})$, S^1.

Example. $\mathbb{Z}$-reflection groups: S_n, dihedral groups, signed permutation matrices.

In the next two lectures, we'll discuss reflection groups and characters of crystals.

8.1. **Reflection groups.** *Dual vector spaces.* Let R be a commutative ring (ie, my favorite example $\mathbb{Z}$), $\mathbb{F}$ be a field (the field of fractions of R) (ie, $\mathbb{Q}$), $\mathbb{K}$ a field containing $\mathbb{F}$ (ie, $\mathbb{R}$ or $\mathbb{C}$ or $\bar{\mathbb{Q}}$).

Let $\mathfrak{h}_{\mathbb{Z}}^*$ be a vector space (over R). $\mathfrak{h}_{\mathbb{Z}}^* = \operatorname{span}\{\omega_1, \dots, \omega_n\}$ where $\omega_1, \dots, \omega_n$ is a basis. $\mathfrak{h}_{\mathbb{Z}}$ is its dual, $\mathfrak{h}_{\mathbb{Z}} = \operatorname{Hom}(\mathfrak{h}_{\mathbb{Z}}, \mathbb{Z})$. It has basis $\alpha_1^\vee, \dots, \alpha_n^\vee$, defined by $\alpha_i^\vee(\omega_j) = \delta_{i,j}$.

Let $G = GL(\mathfrak{h}_{\mathbb{Z}}^*)$ which we think of as $GL_n(\mathbb{Z})$ ($G \subset GL_n(\mathbb{F})$). G acts on $\mathfrak{h}_{\mathbb{Z}}^*$ ($g\omega_i = \sum_{j=1}^n g_{j,i}\omega_j$).

Write $\langle \mu, \lambda^\vee \rangle = \lambda^\vee(\mu)$, for $\mu \in \mathfrak{h}_{\mathbb{Z}}^*$, $\lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}$. G acts on $\mathfrak{h}_{\mathbb{Z}}$ by

$$\langle g\mu, \lambda^\vee \rangle = \left\langle \mu, g^{-1}\lambda^\vee \right\rangle$$

for $\mu \in \mathfrak{h}_{\mathbb{Z}}^*$, $\lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}$. Note that $G \neq GL(\mathfrak{h}_{\mathbb{Z}})$, in terms of matrices g acting on $\mathfrak{h}_{\mathbb{Z}}$ by the matrix $g^\vee = (g^{-1})^t$.

Definition. A *reflection* is $s \in GL(\mathfrak{h}_{\mathbb{Z}}^*)$ such that, in $GL_n(\hat{\mathbb{F}})$, s is conjugate to $\begin{pmatrix} \xi & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}$, where $\xi \in \hat{\mathbb{F}}$, $\xi \neq 1$.

s acts on $\mathfrak{h}_{\mathbb{Z}}$ by $s^\vee$, which is also a reflection. We can write

$$\mathfrak{h}_{\mathbb{C}}^* = \mathfrak{h}_{\mathbb{C}}^{\alpha^\vee} \oplus \mathbb{C}\alpha \quad \text{and} \quad \mathfrak{h}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}}^{\alpha} \oplus \mathbb{C}\alpha^\vee,$$

where

$$\begin{aligned}
\mathfrak{h}_{\mathbb{C}}^{\alpha^\vee} &= (\mathfrak{h}_{\mathbb{C}}^*)^s = \{\mu \in \mathfrak{h}_{\mathbb{C}}^* | s\mu = \mu\} \quad (1 \text{ eigenspace of } s \text{ on } \mathfrak{h}_{\mathbb{C}}^*),\\
\mathbb{C}\alpha &= \{\mu \in \mathfrak{h}_{\mathbb{C}}^* | s\mu = \xi\mu\} \quad (\xi \text{ eigenspace of } s \text{ on } \mathfrak{h}_{\mathbb{C}}^*),\\
\mathfrak{h}_{\mathbb{C}}^{\alpha} &= (\mathfrak{h}_{\mathbb{C}})^s = \{\lambda^\vee \in \mathfrak{h}_{\mathbb{C}} | s\lambda^\vee = \lambda^\vee\} \quad (1 \text{ eigenspace of } s^\vee \text{ on } \mathfrak{h}_{\mathbb{C}}),\\
\mathbb{C}\alpha^\vee &= \{\lambda^\vee \in \mathfrak{h}_{\mathbb{C}} | s\lambda^\vee = \xi^{-1}\lambda^\vee\} \quad (\xi^{-1} \text{ eigenspace of } s \text{ on } \mathfrak{h}_{\mathbb{C}}).
\end{aligned}$$

Choose α and $\alpha^\vee$ so that $1 - \langle \alpha, \alpha^\vee \rangle = \xi$. Then

$$s\mu = \mu - \langle \mu, \alpha^\vee \rangle \alpha \quad \text{and} \quad s^{-1}\lambda^\vee = \lambda^\vee - \langle \lambda^\vee, \alpha \rangle \alpha^\vee \tag{1}$$

You should check that $\mathfrak{h}_{\mathbb{C}}^{\alpha^\vee} = (\mathfrak{h}_{\mathbb{C}}^*)^s = \{\mu \in \mathfrak{h}_{\mathbb{C}}^* | \langle \mu, \alpha^\vee \rangle = 0\}$. If $\mu \in \mathfrak{h}_{\mathbb{C}}^{\alpha^\vee}$ then equation (1) implies

$$\begin{aligned}
&s\mu = \mu - \langle \mu, \alpha^\vee \rangle \alpha = \mu - 0 = \mu \quad \text{and}\\
&s\alpha = \alpha - \langle \alpha, \alpha^\vee \rangle = (1 - \langle \alpha, \alpha^\vee \rangle)\alpha = \xi\alpha.
\end{aligned}$$

8.2. Weyl groups = $\mathbb{Z}$-reflection groups = crystallographic reflection groups.

Let $\mathfrak{h}_{\mathbb{Z}}^*$ be a $\mathbb{Z}$-vector space.

Definition. A *Weyl group* is a finite subgroups W_0 of $GL(\mathfrak{h}_{\mathbb{Z}}^*)$ which is generated by reflections. Let R^+ be an index set so that $s_\alpha, \alpha \in R^+$ are the reflections in W_0. (R^+ is the set of positive roots.)

Example. (Type GL_n). $\mathfrak{h}_{\mathbb{Z}}^* = \text{span}\,\{\epsilon_1, \dots, \epsilon_n\}$ and $W_0 = S_n$ acts by permutations of $\epsilon_1, \dots, \epsilon_n$. The reflections in S_n are

$$s_{i,j} = s_{\epsilon_i^\vee - \epsilon_j^\vee} = \begin{pmatrix}
1 &&&&&&&&&& \\
& \ddots &&&&&&&&& \\
&& 1 &&&&&&&& \\
&&& 0 &&&& 1 &&& \\
&&&& 1 &&&&&& \\
&&&&& \ddots &&&&& \\
&&&&&& 1 &&&& \\
&&& 1 &&&& 0 &&& \\
&&&&&&&& 1 && \\
&&&&&&&&& \ddots & \\
&&&&&&&&&& 1
\end{pmatrix},$$

for $1 \le i < j \le n$. We have

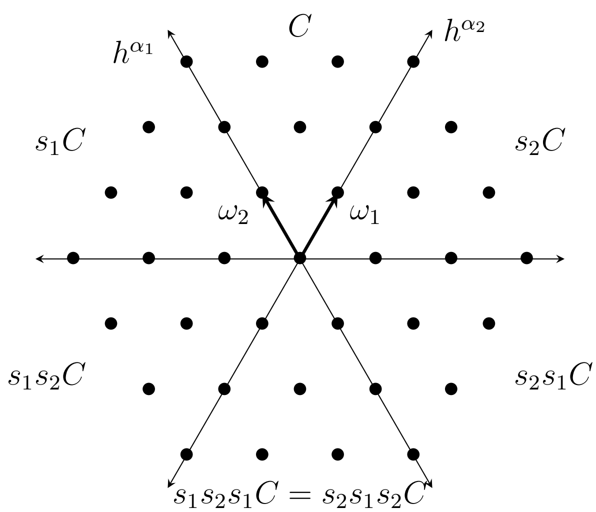
$$R^+ = \{(i,j) | 1 \le i < j \le n\} = \{\epsilon_i^\vee - \epsilon_j^\vee | 1 \le i < j \le n\}.$$

Note: $\det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$ and $s_{i,j}^2 = 1 \Leftrightarrow (s_{i,j}+1)(s_{i,j}-1) = 0$. This is generally true: If $g \in GL_n(\mathbb{Z})$ then $\det(g) \in \mathbb{Z}$ is invertible so $\det(g) = \pm 1$; so, in a Weyl groups, $\xi = -1$.

Example. (Type SL_3) $\mathfrak{h}_{\mathbb{Z}}^* = \text{span}\,\{\omega_1, \omega_2\}$ and

$$W_0 = \left\langle s_i, s_2 | s_1^2 + s_2^2 = 1, s_1 s_2 s_1 = s_2 s_1 s_2 \right\rangle$$

where s_1 is reflection in $\mathfrak{h}^{\alpha_1^\vee}$ and s_2 is reflection in $\mathfrak{h}^{\alpha_2^\vee}$.



This lattice is in $\mathfrak{h}_{\mathbb{R}}^*$ (lattice means $\mathbb{Z}$-vector space).

C is a choice of fundamental region for the action of W_0 on $\mathfrak{h}_{\mathbb{R}}^*$ which is the $\mathbb{R}$-span of ω_1, ω_2. Ah, let's do this in general. $\mathfrak{h}^{\alpha_1^\vee}, \ldots, \mathfrak{h}^{\alpha_n^\vee}$ are the walls (hyperplanes) bounding this fundamental region C. The simple reflections in W_0 are $s_1, \ldots, s_n$, the reflections in $\mathfrak{h}^{\alpha_1^\vee}, \ldots, \mathfrak{h}^{\alpha_n^\vee}$

$W_0 \longleftrightarrow \{$fundamental regions for the action of W on $\mathfrak{h}_{\mathbb{R}}^*\}$.

8.3. **Towards characters.** Let $X = \{X^\mu | \mu \in \mathfrak{h}_{\mathbb{Z}}^*\}$ with $X^\mu X^\nu = X^{\mu+\nu}$. This is the same group as $\mathfrak{h}_{\mathbb{Z}}^*$ except written multiplicatively.

$$\mathbb{C}[X] = \text{span}\,\{X^\mu | \mu \in \mathfrak{h}_{\mathbb{Z}}^*\} = \mathbb{C}[X^{\pm\omega_1}, \ldots, X^{\pm\omega_n}],$$

since $X^\mu = X^{\mu_1\omega_1 + \cdots + \mu_n\omega_n} = X^{\mu_1\omega_1} \cdots X^{\mu_n\omega_n} = (X^{\omega_1})^{\mu_1} \cdots (X^{\omega_n})^{\mu_n}$, for $\mu = \mu_1\omega_1 + \cdots + \mu_n\omega_n$, $\mu_i \in \mathbb{Z}$.

W_0 acts on $\mathbb{C}[X]$ by

$$wX^\mu = X^{w\mu}$$

for $\mu \in \mathfrak{h}_{\mathbb{Z}}^*$, $w \in W_0$.

There are two 1-dimensional representations of W_0:

$$\begin{aligned} W_0 &\to \mathbb{C}^* \\ w &\mapsto 1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} W_0 &\to \mathbb{C}^* \\ w &\mapsto \det w \end{aligned}$$

The ring of *symmetric functions* is

$$\mathbb{C}[X]^{W_0} = \{f \in \mathbb{C}[X] | wf = f \text{ for all } w \in W_0\}$$

The vector space of *determinant-symmetric functions* is

$$\mathbb{C}[X]^{\det} = \{f \in \mathbb{C}[X] | wf = \det(w) f \text{ for all } w \in W_0\}.$$

Example. (Type GL_n) $\mathfrak{h}_{\mathbb{Z}}^* = \text{span}\{\epsilon_1, \dots, \epsilon_n\}$ and $W_0 = S_n$.

$\mathbb{C}[X] = \mathbb{C}[X^{\pm\epsilon_1}, \dots X^{\pm\epsilon_n}] = \mathbb{C}[x_1^{\pm 1}, \dots x_n^{\pm 1}]$ where $x_i = X^{\epsilon_i}$.

For example, the polynomial

$$x_1^2 x_2^{-1} x_3^4 + x_1^{-1} x_2^2 x_3^4 + x_1^4 x_2^{-1} x_3^2 + x_1^2 x_2^4 x_3^{-1} + x_1^{-1} x_2^4 x_3^2 + x_1^4 x_2^2 x_3^{-1}$$

is symmetric; the polynomial

$$x_1^2 x_2^{-1} x_3^4 - x_1^{-1} x_2^2 x_3^4 - x_1^4 x_2^{-1} x_3^2 - x_1^2 x_2^4 x_3^{-1} + x_1^{-1} x_2^4 x_3^2 + x_1^4 x_2^2 x_3^{-1}$$

is determinant-symmetric.

In general,

$$\sum_{w \in S_n} \det w^{-1} \cdot w(x_1^{\mu_1} \dots x_n^{\mu_n}) = \sum_{w \in S_n} \det w^{-1} \cdot x_{w(1)}^{\mu_1} \cdots x_{w(n)}^{\mu_n} = \det x_i^{\mu_j}$$

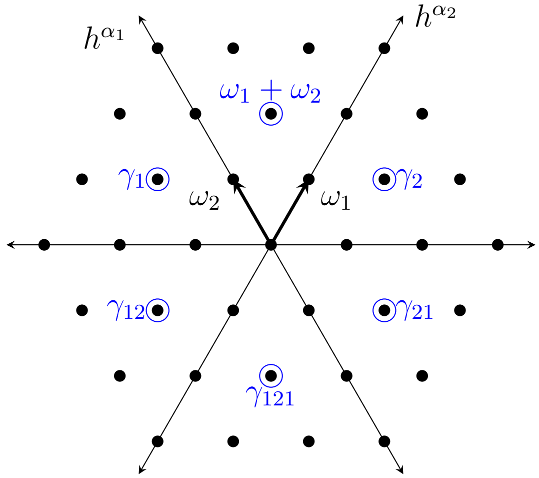
is in $\mathbb{C}[X]^{\det}$.

For example, the Vandermonde determinant:

$$\sum_{w \in W_0} \det w^{-1} = \det \begin{pmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & x_2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ x_n^{n-1} & x_n^{n-2} & \cdots & x_n & 1 \end{pmatrix}$$

Definition. If $\mu \in \mathfrak{h}_{\mathbb{Z}}^*$ then the *orbit sum*, the *monomial symmetric function*, is $m_\mu := \sum_{\gamma \in W_0 \mu} X^\gamma$, and $m_{w\mu} = m_\mu$ for $w \in W_0$.

For example, the orbit of $\omega_1 + \omega_2$ in the SL_3-type example is in blue here:



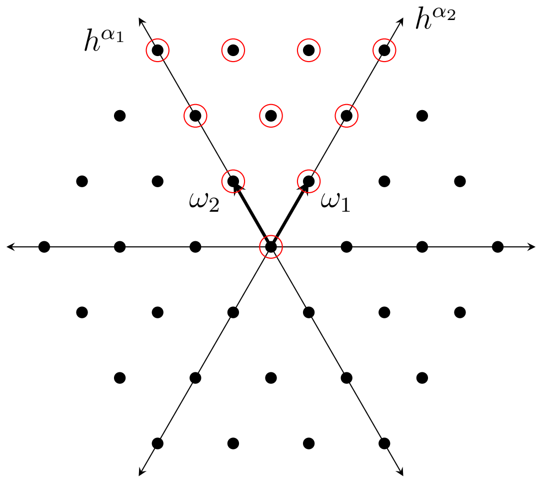
And so we have $m_{\omega_1+\omega_2} = X^{\omega_1+\omega_2} + X^{\gamma_1} + X^{\gamma_2} + X^{\gamma_{12}} + X^{\gamma_{21}} + X^{\gamma_{121}}$.

Definition. The *dominant integral weights* are the elements of

$$P^+ = \mathfrak{h}^*_{\mathbb{Z}} \cap \bar{C},$$

where $\bar{C}$ is the closure of C. These are (distinct) representatives of the W_0 orbits on $\mathfrak{h}^*_{\mathbb{Z}}$.

So, for example, we circle in red the dominant integral weights of our SL_3-type example:



The point is, the m_μ, $\mu \in P^+$ form a basis of $\mathbb{C}[X]^{W_0}$.

If $\mu \in \mathfrak{h}_{\mathbb{Z}}^*$ define

$$a_\mu = \sum_{w \in W_0} \det w^{-1} \cdot X^{w\mu} \in \mathbb{C}[X]^{\det}.$$

If $v \in W_0$, then

$$a_{v\mu} = \sum_{w \in W_0} \det w^{-1} \cdot X^{wv\mu} = \sum_{w \in W_0} \det v \det (wv)^{-1} \cdot X^{wv\mu} = \det v \cdot a_\mu.$$

So, for example in SL_3, $a_{\omega_2 - \omega_1} = \det s_1 \cdot a_{\omega_1} = -a_{\omega_1}$.

Another example: $a_{s_2\omega_1} = \det(s_2) a_{\omega_1} = -a_{w_1}$ but on the other hand $s_2\omega_1 = \omega_1$, so $a_{s_2\omega_1} = a_{\omega_1}$, and thus $a_{\omega_1} = 0$.

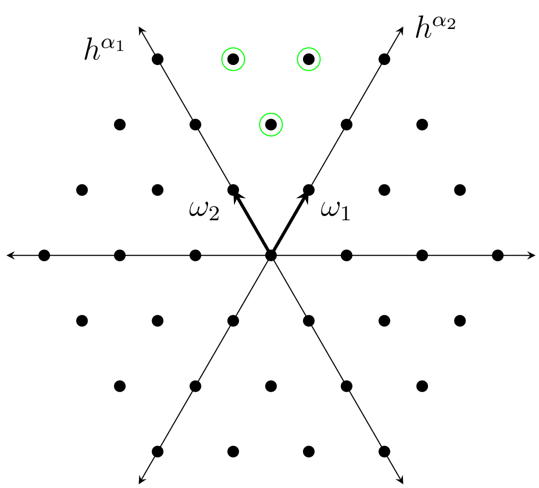
In general if μ is on a wall, ie $s_\alpha \mu = \mu$ for some reflection, then $a_\mu = 0$.

Definition. The *strictly dominant weights* are elements of

$$P^{++} = \mathfrak{h}_{\mathbb{Z}}^* \cap C$$

(C does not include the walls).

For example,



So we see that the a_μ, $\mu \in P^{++}$ are a basis of $\mathbb{C}[X]^{\det}$; and recall that the $m_\mu, \mu \in P^+$ are a basis of $C[X]^{W_0}$.

As sets (or semigroups), P^+ is isomorphic to P^{++}, via $\lambda \mapsto \lambda + \rho$ where ρ is the vertex of the cone P^{++}.

The m_λ, $\lambda \in P^+$ are a basis of $\mathbb{C}[X]^{W_0}$ (these are bosonic); $a_{\lambda+\rho}$, $\lambda \in P^+$ are a basis of $\mathbb{C}[X]^{\det}$ (fermionic because they're alternating).

We're seeing a version of the Boson-Fermion correspondence: $\mathbb{C}[X]^{W_0}$ is isomorphic to $\mathbb{C}[X]^{\det}$, via $f \mapsto a_\rho f$. We saw only a shadow of this, the set version, today: P^+ and P^{++} are isomorphic.