REPRESENTATION THEORY

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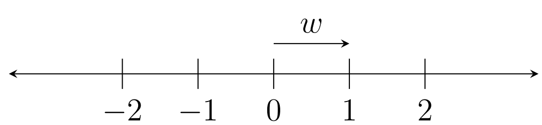
Abstract. Notes from Arun Ram's 2008 course at the University of Melbourne.

6. Week 6

This lecture is given by Richard Brak. He's a combinatorist, and says today's lecture will be very combinatorial.

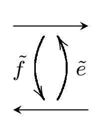
6.1. $\mathfrak{sl}_2$-Crystals.



Definition. Start with $B(\square) = \{\longrightarrow, \longleftarrow\}$, representing unit vectors, or a step to the right or left, and a couple of operators $\tilde{e}$ and $\tilde{f}$ (root operators or Kashiwara operators) given by

$$\tilde{e}(\longleftarrow) = \longrightarrow, \qquad \tilde{f}(\longleftarrow) = 0$$
$$\tilde{e}(\longrightarrow) = 0, \qquad \tilde{f}(\longrightarrow) = \longleftarrow$$

We represent this pictorally as



The tensor product of two sets of paths will just be the cartesian product of the sets, with concatenation as the product on paths; so for

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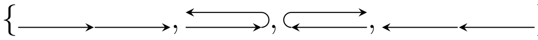
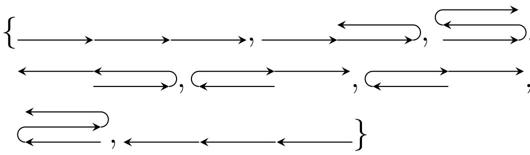
example

$B(\square) \otimes B(\square) = \{$  $\}$

and

$B(\square) \otimes B(\square) \otimes B(\square) = \{$  $\}$

Our convention is to read such a path bottom to top, or, if representing $\longrightarrow$ by 1 and $\longleftarrow$ by 2, as a word going right to left.

If B_1 and B_2 are $\mathfrak{sl}_2$ crystals, then $\tilde{f}$ acts on

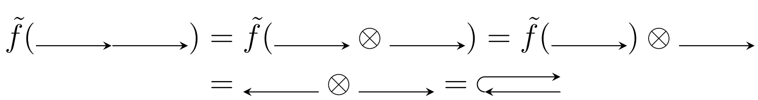
$$B_1 \otimes B_2 = \{p \otimes q | p \in B_1, q \in B_2\}$$

by

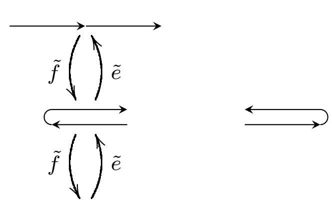
$$\tilde{f}(p \otimes q) = \begin{cases} (\tilde{f}p) \otimes q & \text{if (last occurence of) leftmost step to the right, in } p \otimes q \text{, is in } p \\ p \otimes (\tilde{f}q) & \text{if (last occurence of) leftmost step to the right, in } p \otimes q \text{, is in } q \\ 0 & \text{no leftmost step to right.} \end{cases}$$

and the action of $\tilde{e}$ is defined by $\tilde{e}b = b'$ if $b = \tilde{f}b'$, 0 otherwise. We will make a more precise definition later.

Let's look at the action of $\tilde{e}$ and $\tilde{f}$ on $B(\square) \otimes B(\square)$. For example,



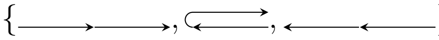
The action of $\tilde{e}$ and $\tilde{f}$ on all of $B(\square) \otimes B(\square)$ are summarized by

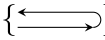


Question. I don't understand why $\tilde{e}$ and $\tilde{f}$ act as zero on this last path.

Answer. There's not a leftmost step to the right of the rightmost point here. There's a better definition of $\tilde{e}$ and $\tilde{f}$ action coming later.

So we've shown that as crystals, $B(\square) \otimes B(\square) = B(\square\square) \sqcup B(\emptyset)$ for

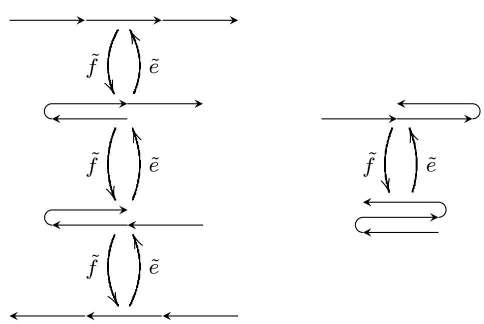
$B(\square\square) = \{$  $\}$

$B(\emptyset) = \{$  $\}$.

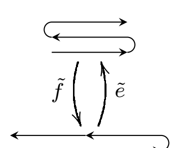
Now we compute

$$\begin{aligned} B(\square)^{\otimes 3} &= (B(\square) \otimes B(\square)) \otimes B(\square) \\ &= B(\square\square) \otimes B(\square)) \sqcup (B(\emptyset) \otimes B(\square)) \end{aligned}$$

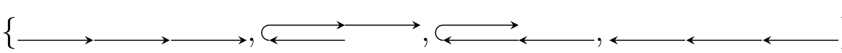
The crystal structure of $B(\square\square) \otimes B(\square)$ is



And the structure of $B(\emptyset) \otimes B(\square)$ is

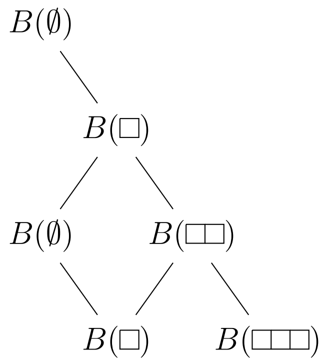


And so $B(\square\square) \otimes B(\square) = B(\square\square\square) \sqcup B(\square)$, $B(\emptyset) \otimes B(\square) \simeq B(\square)$, for

$B(\square\square\square) = \{$  $\}$.

Hence $B^{\otimes 3} \simeq B(\square\square\square) \sqcup B(\square) \sqcup B(\square)$.

So far, we've seen the Bratelli diagram for the decomposition of $B^{\otimes k}$ is



Definition. A *crystal* is a subset of $B^{\otimes \ell}$ close under the action of $\tilde{e}$ ad $\tilde{f}$.

Definition. The *crystal graph* of a crystal B is the graph with vertices B and edges between p and $\tilde{f}p$.

Definition. A crystal is *irreducible* if its crystal graph is connected.

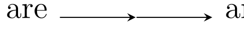
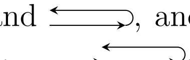
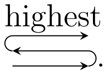
Definition. Let B be a crystal. The character of B is $\mathrm{ch}(B) = \sum_{p \in B} x^{\mathrm{wt}(p)}$ where wt(p) is the coordinate of the end point of p.

Definition. A *highest weight* path is a path which always stays to the right of 0.

In our examples we've seen that if p is a highest weight path then $\tilde{e}p = 0$.

Some character calculations:

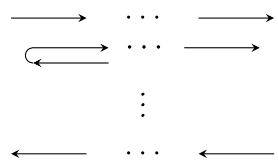
$$\begin{aligned}
\mathrm{ch}(B(\square)) &= x + x^{-1} \\
\mathrm{ch}(B(\square\square)) &= x^2 + 1 + x \\
\mathrm{ch}(B(\emptyset)) &= 1 \\
\mathrm{ch}(B(\square)^{\otimes 2}) &= (x + x^{-1})^2 = x^2 + 2 + x^{-2} = (x^2 + 1 + x) + (1) \\
\mathrm{ch}(B(\square\square\square)) &= x^3 + x + x^{-1} + x^{-3} \\
\mathrm{ch}(B(\square)^{\otimes 3}) &= (x + x^{-1})^3 = (x^3 + x + x^{-1} + x^{-3})^2 + (x + x^{-1}) + (x + x^{-1})
\end{aligned}$$

The highest weight paths of $B^{\otimes 2}$ are and , and the highest weight paths of $B^{\otimes 3}$ are , , and .

Theorem 6.1 (Classification of irreducible $\mathfrak{sl}_2$-crystals).

(1) *The irreducible $\mathfrak{sl}_2$-crystals are*

$$B(\underbrace{\square\square\cdots\square}_{k}) = \left\{ \right.$$



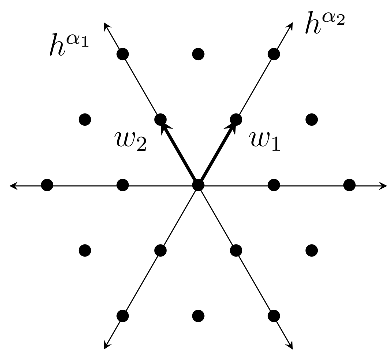
$$\left. \right\}$$

with $ch(B(\underbrace{\square\square\cdots\square}_{k})) = x^k + x^{k-2} + \cdots + x^{-(k-2)} + x^{-k}$.

(2) *Every crystal is a disjoint union of irreducible crystals*

(3) *Each irreducible crystal B has a unique highest weight path, and* $B \simeq B(\underbrace{\square\square\cdots\square}_{k})$ *if p ends at k.*

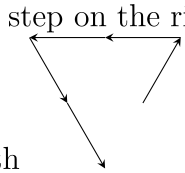
6.2. **$\mathfrak{sl}_3$-Crystals.**



and operators $\tilde{e}_1$, $\tilde{e}_2$, $\tilde{f}_1$ and $\tilde{f}_2$.

Example. $B(\square) = \{w_1 =$  $, w_2 - w_1 =$ $, -w_2 =$  $\}$ and for ease of notation we write 1 for w_1, 2 for $w_2 - w_1$ and 3 for $-w_2$.

The action of $\tilde{e}_i$ and $\tilde{f}_i$ on a path in $B(\square)^{\otimes k}$ is defined as follows:

(1) Write the path as a word, with first step on the right (and last step leftmost). For example, the path  is 33221, and the path 13132312223 is, well, you can figure it out.

(2) Take the subword ρ_2 consisting only of the numbers (i) and $(i+1)$ (eg for $i = 2$ the subword of 13132312223 is 33232223, ie

we delete all 1s). Geometrically this is projection onto one of the root axes.

(3) Recursively delete adjacent pairs $(i+1),(i)$ to leave another subword $\hat{\rho}_2$ (in the running example we are left with 23). $\hat{\rho}_2$ is always of the form $(i)^r(i+1)^s$.

(4) $\tilde{f}_i((i)^r(i+1)^s) = \begin{cases} (i)^{r-1}(i+1)^{s+1} & \text{if } r \geq 1 \\ 0 & \text{otherwise} \end{cases}$

$\tilde{e}_i((i)^r(i+1)^s) = \begin{cases} (i)^{r+1}(i+1)^{s-1} & \text{if } r \geq 1 \\ 0 & \text{otherwise} \end{cases}$

(5) Now change the original word by changing its subword $\hat{\rho}_2$ as above.

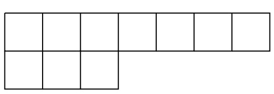
Example. $\tilde{f}_2(13132312223) = 13132312233$

Here $B^{\otimes 2}$ has 9 paths, of two steps each; its crystal structure (as shown in Figure 1) implies $B^{\otimes 2} = B(\square\square) \sqcup B(\begin{smallmatrix}\square\\\square\end{smallmatrix})$.

The points of the positive/dominant chamber,

$$p^+ = \{kw_1 + \ell w_2 | k, \ell \in \mathbb{Z}_{\geq 0}\},$$

are in bijection with partitions with ≤ 2 rows; $kw_1+\ell w_2$ corresponds to (ℓ, k) which can represent a Young diagram, which has a 2 by ℓ block, with a 1 by k block tacked on. For example, if $\ell = 3$ and $k = 4$ we have



And we can define characters for $\mathfrak{sl}_3$-crystals too; we have

$$\begin{aligned} \mathrm{ch}(B^{\otimes 2}) =& (x_1+x_2+x_3)^2 \\ =& (x_1^2 + x_1x_2 + x_3x_1 + x_2^2 + x_3x_2 + x_3^2) + (x_1x_2 + x_1x_3 + x_2x_3) \end{aligned}$$

$$\mathrm{ch}(B(\square\square)) = \sum_{1 \leq i \leq j \leq 3} x_i x_j$$

$$\mathrm{ch}(B(\begin{smallmatrix}\square\\\square\end{smallmatrix})) = \sum_{1 \leq i < j \leq 3} x_i x_j$$

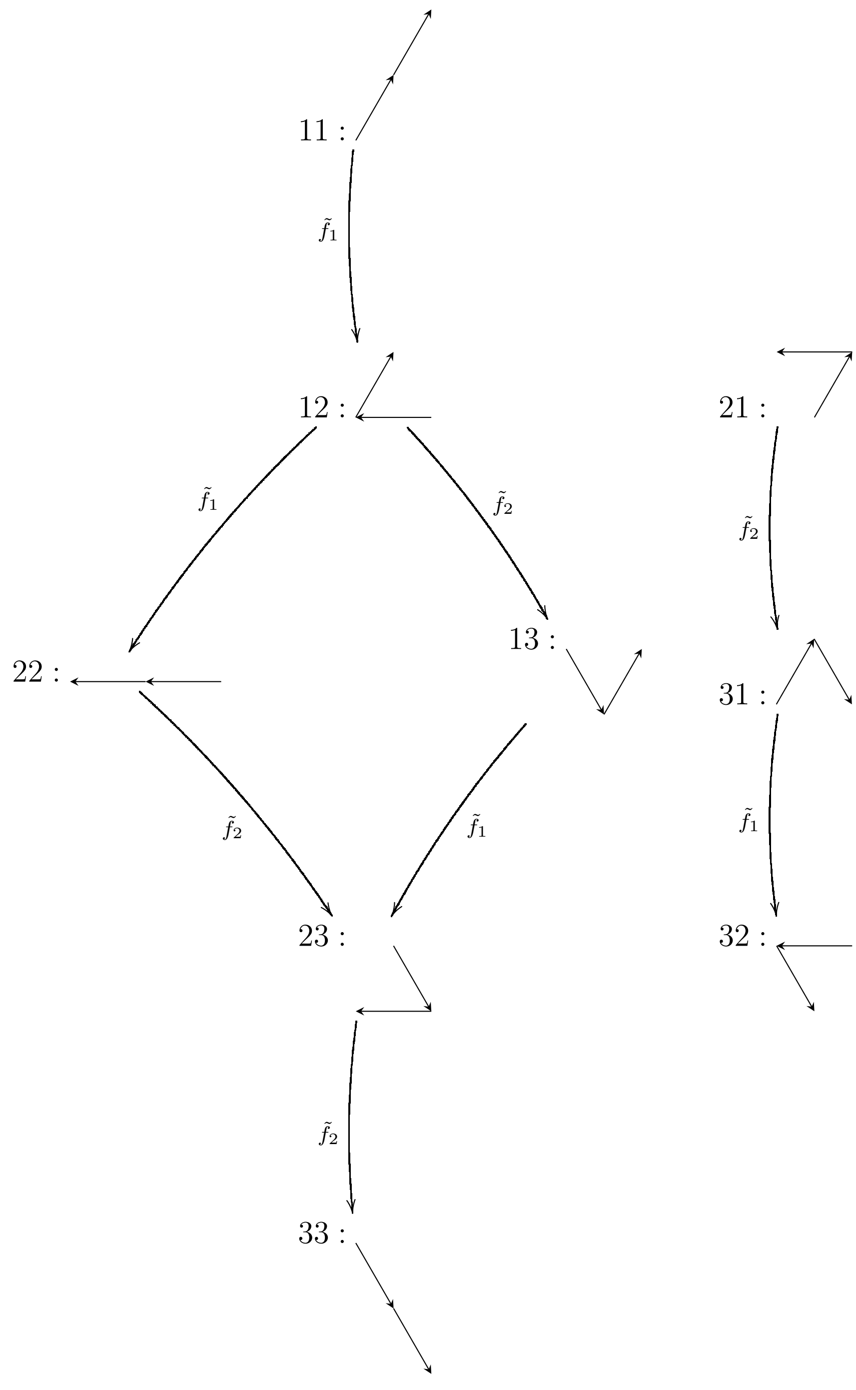


FIGURE 1. The crystal structure of $B^{\otimes 2}$ for $\mathfrak{sl}_3$