REPRESENTATION THEORY

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ABSTRACT. Notes from Arun Ram's 2008 course at the University of Melbourne.

1. Week 1

2. Week 2

Theorem 2.1 (Artin-Wedderburn)**.** *(Almost) every algebra A is semisimple, $A = \oplus_{\lambda \in \hat{A}} M_{d_\lambda}(\mathcal{C})$*

Counter-example. Last week we has the counter-example that

$$\left\{ \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

is not semisimple.

However, there is a problem: this is not an algebra (no identity). We can try to fix this:

$$\left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\}$$

is not semisimple, but the proof is different from the proof we used last week.

2.1. **Remark about generators and relations.**

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Definition. The general Temperley-Lieb algebra TL_k is:

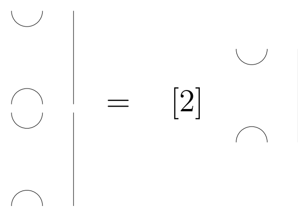
$$TL_k = \text{span}\left\{ \begin{array}{l} \text{noncrossing (planar) diagrams with} \\ k \text{ top dots and } k \text{ bottom dots} \end{array} \right\}$$

with the product

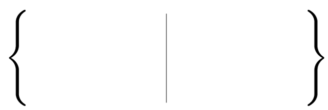
$$b_1 b_2 = (q+q^{-1})^{\#\text{ of internal loops}}(b_1 \text{ on top of } b_2)$$

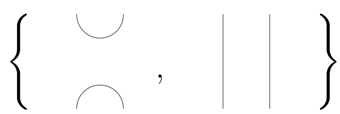
(ie, blob= $(q+q^{-1}) = [2]$.)

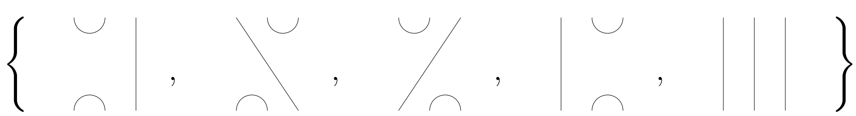
Example.



Example.

$TL_1 =$ span 

$TL_2 =$ span 

$TL_3 =$ span 

$TL_4 =$ span 

These have dimensions 1,2,5,14, ...which are the Catalan numbers.

Definition. Let $e_i = \Big| \ldots \Big| \overset{\smile}{\underset{\frown}{\ }} \Big| \ldots \Big|$, $i = 1, 2, \ldots, k-1$.

Theorem 2.2. *TL_k is presented by generators $e_i, \ldots, e_{k-1}$ and relations*

$$e_i^2 = (q + q^{-1})e_i \quad \textit{and} \quad e_i e_{i\pm1} e_i = e_i$$

Remark. It's not possible to define an algebra except by generators and relations. Whenever we want to show that an algebra, defined in terms of generators A and relations A, is presented by generators B and relations B, what we really need to do is show:

(1) generators A can be written in terms of generators B
(2) relations A can be derived from relations B
(3) generators B can be written in terms of generators A
(4) relations B can be derived from relations A

Proof. In the definition of Temperley-Lieb, let generators A be {noncrossing (planar) diagrams with k top dots and k bottom dots}, and relations A be $\{b_1 b_2 = (q+q^{-1})^{\#\text{ of internal loops}}(b_1 \text{ on top of } b_2)\}$. Now let generators B be $\{e_i\}$, and relations B be $\{e_i^2 = (q + q^{-1})e_i$ and $e_i e_{i\pm1} e_i = e_i\}$.

(3) and (4) are easy in this case; (1) and (2) are the hard parts. ☐

2.2. **Traces.**

Definition. Let A be an algebra. A *trace* on A is a linear transformation $t : A \to \mathbb{C}$ such that

$$t(a_1 a_2) = t(a_2 a_1) \quad \text{for } a_1, a_2 \in A.$$

Define $\langle , \rangle : A \otimes A \to \mathbb{C}$ by

$$\langle a_1, a_2 \rangle = t(a_1 a_2) \quad \text{for } a_1, a_2 \in A.$$

Note:

$$\langle a_1, a_2 \rangle = \langle a_2, a_1 \rangle \quad \text{and} \quad \langle a_1 a_2, a_3 \rangle = \langle a_1, a_2 a_3 \rangle .$$

Definition. The *radical* of $\langle , \rangle$ is

$$\text{Rad}(\langle , \rangle) = \{r \in A |\, \langle r, a \rangle = 0 \text{ for all } a \in A\}$$

Homework. $\text{Rad}(\langle , \rangle)$ is an ideal of A (ie if $r \in \text{Rad}(\langle , \rangle)$ and $a \in A$ then $ra, ar \in \text{Rad}(\langle , \rangle)$.

Definition. The trace t of the form $\langle , \rangle$ is *nondegenerate* if

$$\mathrm{Rad}(\langle , \rangle) = 0.$$

Definition. Let B be a basis of A, $B = \{b_1, \dots, b_n\}$. The *dual basis* to B with respect to $\langle , \rangle$ is $B^* = \{b_1^*, \dots, b_n^*\}$ such that

$$\left\langle b_i, b_j^* \right\rangle = \delta_{i,j}.$$

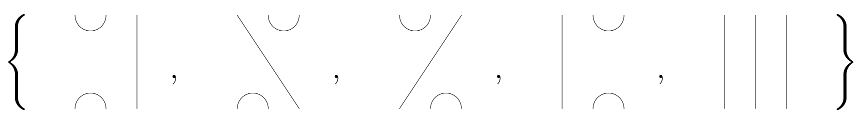
Definition. The *Gram matrix* of $\langle , \rangle$ is

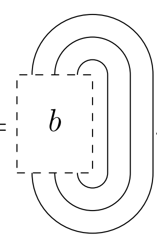
$$G = (\langle b_i, b_j \rangle)_{b_i, b_j \in B}.$$

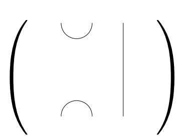
Homework. The dual basis exists iff the Gram matrix is invertible iff $\det(G)$ is inveritble in $\mathbb{C}$ iff $Rad(\langle , \rangle) = 0$.

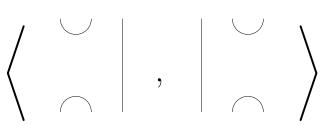
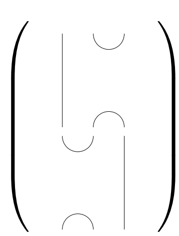
Let A be an algebra with a nondegenerate trace t. Let B be your basis of A.

Example. Let $A = TL_3$,

$B =$  .

My favorite trace is $t(b) = [2]^{\#\text{ of loops in } cl(b)}$ where $cl(b) =$  .

So, for example, t  $= [2]^2$ and

 $= t$  $= [2]$

2.3. **Commuting operators.** Again: Let A be an algebra with a nondegenerate trace t. Let B be your basis of A. Let B^* be the dual basis. Let M, N be A-modules. Recall

$$\rho_M : A \to End(M) \qquad \text{and } \rho_N : A \to End(N)$$
$$a \mapsto a_M \qquad\qquad a \mapsto a_N$$

Then

$$\mathrm{Hom}_A(M,N) = \left\{ \phi : M \to N \middle| \begin{array}{l} \phi \text{ is a morphism of vector spaces and} \\ \phi(am) = a\phi(m), \text{ for } a \in A, m \in M \end{array} \right\}$$
$$= \{\phi \in Hom(M,N) | \phi a_M = a_N \phi\}$$

Definition. The *A-endomorphisms of M* are

$$\mathrm{End}_A(M) := \{\phi \in \mathrm{End}(M) | \phi a_M = a_M \phi \text{ for } a \in A\}$$

where $\mathrm{End}(M) = \mathrm{Hom}(M, M)$. Or we might just write

$$\mathrm{End}_A(M) = \{\phi \in \mathrm{End}(M) | \phi a = a\phi \text{ for } a \in A\}$$

Now, let $\phi : M \to N$ be a vector space homomorophism. Define $[\phi] : M \to N$ by

$$[\phi] = \sum_{b \in B} b\phi b^*.$$

(and check that if $m \in M$, $[\phi]m = \sum_b b\phi b^* m \in N$).

Claim. $[\phi] \in Hom_A(M,N)$.

Proof. Let $a \in A, m \in M$.

$$\begin{aligned} a[\phi]m &= \sum_{b \in B} ab\phi b^* m = \sum_{b \in B} \sum_{c \in B} \langle ab, c^* \rangle c\phi b^* m \\ &= \sum_{b,c \in B} c\phi \langle ab, c^* \rangle b^* m = \sum_{b,c \in B} c\phi \langle c^* a, b \rangle b^* m = \sum_{c \in B} c\phi c^* am \\ &= [\phi]am. \end{aligned}$$

□

Homework. Show that $[\phi]$ does not depend on the choice of B.

Game. You give me $\phi : M \to N$ and I make $[\phi] \in Hom_A(M,N)$.

Detour: Schur's lemma. Suppose $[\phi] \in \mathrm{Hom}_A(M,N)$ and suppose M and N are simple. Then $\ker[\phi]$ and $\mathrm{im}[\phi]$ are submodules of M

and N respectively. So $\ker[\phi] = 0$ or $\ker[\phi] = M$ and $\mathrm{im}[\phi] = 0$ or $\mathrm{im}[\phi] = N$. So either $[\phi]$ is zero, or $[\phi]$ is injective and surjective, ie an isomorphism. If $M \simeq N$ then $[\phi] \in \mathrm{End}_A(M)$. Since $\mathcal{C}$ is an algebraically closed field $[\phi]$ has an eigenvalue λ. Then $[\phi] - \lambda \in \mathrm{End}_A(M)$. So $[\phi] - \lambda = 0$ or $[\phi] - \lambda$ is an isomorphism. Since $det([\phi] - \lambda) = 0, [\phi] - \lambda = 0$, ie $[\phi] = \lambda$. We've just proved

Theorem 2.3 (Schur's Lemma)**.** *Suppose $[\phi] \in Hom_A(M, N)$ and suppose M and N are simple. Then either $[\phi] = 0$ or $[\phi] = \lambda$ for some λ. In particular, if M is simple, then*

$$End_A(M) = \mathcal{C}.$$

Definition. Let A be an algebra. Let M be an A-module. The *commutant* or *centralizer algebra* of M is $End_A(M)$.

General question: How are A and $End_A(M)$ related?

2.4. **Regular representation.**

Definition. Let A be an algebra. The *regular representation* of A is A with A-action given by left multiplication. Then

$$\rho_A : A \to End(A)$$
$$a \mapsto a_A$$

is injective, since $a \cdot 1 = a$ implies $\ker \rho_A = 0$.

Therefore, elements of A "are" matrices. (You may have thought that Temperley-Lieb was diagrams, but it turns out it's nothing more than a bunch of 5-by-5 matrices.)

Let $t : A \to \mathcal{C}$ be the *trace of the regular representation*

$$t(a) := Tr(a_A)$$

Theorem 2.4. *(Maschke's theorem) Let A be an algebra such that the trace of the regular representation is nondegenerate (note that finite dimensionality has already entered here – infinite matrices might not have traces). Then every A-module M is completely decomposable, ie*

$$M = A^\lambda \oplus A^\mu \oplus \cdots$$

where $A^\lambda, A^\mu, \ldots$ are simple modules.

Proof. Let M be an A-module. If M is simple, we're done.

Otherwise let N be a submodule of M. N has basis $\{n_1, \dots, n_r\}$ and M has basis $\{n_1, \dots, n_r, m_1, \dots, m_s\}$.

Define a map $\phi : M \to M$ by $\phi(n_i) = n_i$ and $\phi(m_j) = 0$. Then $\phi(n) = n$ for $n \in N$ and $\phi^2 = \phi$, $\text{im}\phi = N$, so ϕ is projection onto N. And $[\phi] \in Hom_A(M, M)$.

If $n \in N$ then

$$[\phi]n = \sum_{b \in B} b\phi b^* n = \sum_{b \in B} bb^* n = n,$$

because

Claim. $\sum_{b \in B} bb^* = 1$

Proof. Let $a \in A$ and consider $\left\langle \sum_{b \in B} bb^*, a \right\rangle = \sum_{b \in B} \langle ab, b^* \rangle = \sum_{b \in B} ab|_b = Tr(a_A) = \langle 1, a \rangle$. □

Next if $m \in M$,

$$\begin{aligned} [\phi]^2 m = [\phi] \sum_{b \in B} b\phi b^* m &= \sum_{b,c \in B} c\phi c^* b\phi b^* m \\ &= \sum_{b,c,\in B} cc^* b\phi b^* m = \sum_{b \in B} b\phi b^* m = [\phi]m. \end{aligned}$$

So, $[\phi]^2 = [\phi]$ and $(1 - [\phi])^2 = \cdots = 1 - [\phi]$ and $M = 1 \cdot M = ([\phi] + 1 - [\phi])M = [\phi]M + (1 - [\phi])M$. Now $[\phi]M$ is a submodule and $(1 - [\phi])M$ is a submodule, and $[\phi]M \cap (1 - [\phi])M = 0$, so M is split.

By induction, we're done. □