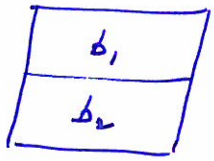
The <u>Temperley-Lieb algebra</u> TL_k is

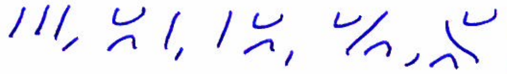
$TL_k = \mathrm{span}\left\{ \begin{array}{l} \text{noncrossing diagrams with} \\ k \text{ top dots and } k \text{ bottom dots} \end{array} \right\}$ (generators A)

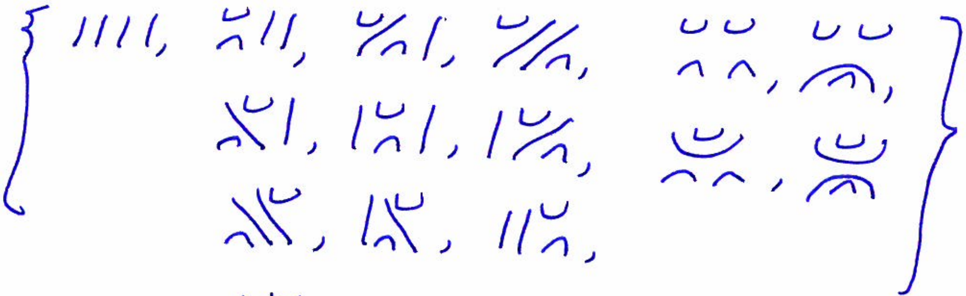
with product

$b_1 b_2 = (q+q^{-1})^{\#\text{ of internal loops}}$ 

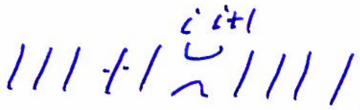
(relations A).

<u>Example:</u> $TL_1 = \mathrm{span}\{ | \}$, $TL_2 = \mathrm{span}\{$  $\}$

$TL_3 = \mathrm{span}\{$  $\}$ and

$TL_4 = \mathrm{span}$ 

Let

$e_i =$  , for $i = 1, \ldots, k-1$. (generators B)

<u>Theorem</u> TL_k is presented by generators $e_1, \ldots, e_{k-1}$ and relations

$e_i^2 = (q+q^{-1})e_i$ and $e_i e_{i\pm1} e_i = e_i$. (relations B)

<u>Proof</u> To show: (a) Generators A can be written in terms of generators B

(b) relations A can be derived from relations B

(c) Generators B can be written in terms of generators A

(d) relations B can be derived from relations A. //

Homework

(1) (a) Define the symmetric group S_k (via permutations).

(b) Let

$$s_i = \; |\,|\,|\,|\overset{i\;\;i+1}{\times}|\,|\,|\,|, \qquad i = 1, 2, \ldots, k-1$$

Show that S_k is presented by generators $s_1, \ldots, s_{k-1}$ and relations

$$s_i^2 = 1 \quad \text{and} \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}.$$

(c) Define the Young lattice and show that it is the Bratelli diagram for the tower

$$\mathbb{C}S_1 \subseteq \mathbb{C}S_2 \subseteq \mathbb{C}S_3 \subseteq \ldots$$

(d) Let

$$m_i = s_{1i} + s_{2i} + s_{3i} + \cdots + s_{i-1,i}, \quad \text{where}$$

$$s_{ij} = \; |\,|\,|\,|\overset{i\qquad j}{\times\!\!\times\!\!\times}|\,|\,| \quad \text{for} \quad 1 \le i < j \le k.$$

Let $m_1 = 0$. Show that

$$m_i m_j = m_j m_i \quad \text{for} \quad 1 \le i < j \le k$$

(e) Show that each irreducible S_k-module S^λ has a basis of simultaneous eigenvectors v_T for $m_1, \ldots, m_k$,

$$\text{i.e.} \quad m_i v_T = c(T(i))\, v_T.$$

(f) Find the eigenvalues $c(T(i))$.

(2) Following the work of R. Block, classify the simple modules of $U_q\mathfrak{sl}_2$, where $U_q\mathfrak{sl}_2$ is the algebra generated by $E, F, K^{\pm 1}$ with relations

$$KK^{-1} = K^{-1}K = \mathbb{1},$$

$$KEK^{-1} = q^2 E, \qquad KFK^{-1} = q^{-2} F,$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

Traces

Let A be an algebra. A *trace* on A is a linear transformation $t: A \to \mathbb{C}$ such that

$$t(a_1a_2) = t(a_2a_1), \quad \text{for } a_1, a_2 \in A.$$

Let $\rho_M: A \to \mathrm{End}(M)$ be a representation of A.
$$a \longmapsto a_M$$

The *character* of M is the trace

$$\begin{aligned}\chi_M: A &\to \mathbb{C} \\ a &\mapsto Tr(a_M)\end{aligned}, \quad \text{where } Tr(a_M) = \sum_{m \in B} am|_m$$

for a basis B of M, with

$$am|_m = \text{coefficient of } m \text{ in } am$$

(expanded in the basis B).

Given a trace $t: A \to \mathbb{C}$ define

$$\langle\,,\,\rangle: A \otimes A \to \mathbb{C} \quad \text{by} \quad \langle a_1, a_2\rangle = t(a_1a_2),$$

for $a_1, a_2 \in A$. Then

$$\langle a_1, a_2\rangle = \langle a_2, a_1\rangle \quad \text{and} \quad \langle a_1a_2, a_3\rangle = \langle a_1, a_2a_3\rangle,$$

for $a_1, a_2, a_3 \in A$. The *radical* of $\langle\,\rangle$ is

$$\mathrm{Rad}(\langle\,,\,\rangle) = \{r \in A \mid \langle r, a\rangle = 0 \text{ for all } a \in A\}.$$

Let $B = \{b_1, \dots, b_n\}$ be a basis of A

The <u>dual basis</u> to B with respect to $\langle , \rangle$ is $B^* = \{b_1^*, \dots, b_n^*\}$ such that

$$\langle b_i, b_j^* \rangle = \delta_{ij}.$$

The <u>Gram matrix</u> of $\langle , \rangle$ is

$$G = \left(\langle b_i, b_j \rangle \right)_{b_i, b_j \in B}.$$

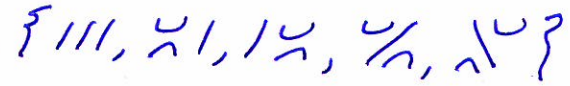
<u>HW</u>: Show that

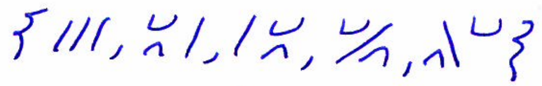
$$\begin{aligned} \mathrm{Rad}(\langle , \rangle) = 0 &\iff G \text{ is invertible} \\ &\iff \det G \text{ is invertible} \\ &\iff \text{The dual basis } B^* \text{ exists} \end{aligned}$$

A <u>nondegenerate</u> <u>trace</u> is a trace $t: A \to \mathbb{C}$ such that $\mathrm{Rad}(\langle\ \rangle) = 0$.

HW: Show that

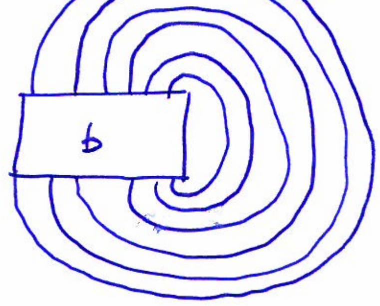
$\mathrm{Rad}(\langle\ \rangle)$ is an ideal of A.

<u>Example</u> $TL_3 = \mathrm{span}$ 

$B =$ 

Define a trace on TL_3 by

$$t(b) = (q+q^{-1})^{\#\text{of cycles in } cl(b)}, \quad \text{where}$$

$cl(b) =$ 

Commuting operators

Let A be an algebra, M an A-module.
The commutant or centralizer algebra is

$$\mathrm{End}_A(M) = \{\varphi \in \mathrm{End}(M) \mid a_M\varphi = \varphi a_M, \text{ for } a \in A\}$$

Recall that

$$\begin{aligned} A &\longrightarrow \mathrm{End}(M) \\ a &\longmapsto a_M \end{aligned}$$ is an algebra homomorphism

Let M and N be simple A-modules and

$\varphi: M \to N$ and A-module homomorphism

(i.e. $\varphi a_M = a_N \varphi$, for $a \in A$). Then

$\ker\varphi$ and $\operatorname{im}\varphi$ are submodules

of M and N, respectively. So $\ker\varphi = 0$ or $\ker\varphi = M$ and $\operatorname{im}\varphi = 0$ or $\operatorname{im}\varphi = N$. So

$\varphi = 0$ or φ is a bijection (and $M \cong N$).

Let λ be an eigenvalue of φ. Then

$\varphi - \lambda \in \mathrm{End}_A(M)$. So $\varphi - \lambda = 0$ or

$\varphi - \lambda$ is invertible. Since $\det(\varphi - \lambda) = 0$, $\varphi - \lambda$ is not invertible. So

$$\varphi = \lambda \cdot Id$$

Schur's Lemma Let M be a simple module. Then $\mathrm{End}_A(M) = \mathbb{C} \cdot id_M$.

Let A be a finite dimensional algebra.
Let $t: A \to \mathbb{C}$ be a nondegenerate trace on A.
Let B be a basis of A and let B^* be the dual basis with respect to $\langle , \rangle$.

<u>Theorem</u> (Maschke's theorem).
(a) Let M, N be A-modules and let
$\varphi: M \to N$ be a vector space morphism.
Then
$$[\varphi] = \sum_{b \in B} b \varphi b^*,$$ is an A-module homomorphism
i.e. $[\varphi] \in \mathrm{Hom}_A(M, N)$.
(b) Assume t is the trace of the regular representation. Every finite dimensional A-module M is completely decomposable

<u>Proof</u> (a) Let $a \in A$. Then
$$a[\varphi] = \sum_{b \in B} ab\varphi b^* = \sum_{b, c \in B} \langle ab, c^* \rangle c \varphi b^*$$
$$= \sum_{b, c \in B} c\varphi \langle ab, c^* \rangle b^* = \sum_{b, c \in B} c\varphi \langle c^* a, b \rangle b^*$$
$$= \sum_{c \in B} c\varphi c^* a = [\varphi] a.$$

<u>HW</u>: Show that φ does not depend on the choice of the basis B.

(b) Let M be a finite dimensional A-module.

Assume $N \subseteq M$ is a nonzero submodule

Let

$\pi : M \to M$ be a vector space homomorphism such that

$$\operatorname{im}\pi = N \quad \text{and} \quad \pi(n) = n, \text{ for } n \in N.$$

(i.e. define $\pi(n_i) = n_i$, ~~for~~ $\pi(m_j) = 0$ for a basis $\{n_1, \ldots, n_r\}$ of N and a basis $\{n_1, \ldots, n_r, m_1, \ldots, m_s\}$ of M)

Then

(ba) $\operatorname{im}[\pi] = N$ and $[\pi]n = n$ for $n \in N$.

(bb) $M = [\pi]M \oplus (1-[\pi])M$, and

$[\pi]M = N$ and $(1-[\pi])M$ are submodules.

Proof of (ba): If $n \in N$

$$[\pi]n = \sum_{b \in B} b\pi b^* n = \Big(\sum_{b \in B} bb^*\Big)n.$$

Since

$$\Big\langle \sum_{b \in B} bb^*, a\Big\rangle = \sum_{b \in B} \langle ab, b^*\rangle = Tr(a_A) = \langle 1, a\rangle$$

for all $a \in A$, it follows that $\sum_{b \in B} bb^* = 1$.

So $[\pi]n = 1 \cdot n = n$.

If $m \in M$ then $[\pi]m = \sum_{b \in B} b\pi b^* m \in N$ since $\pi b^* m \in N$ and N is a submodule.

Proof of (bb). If $m \in M$ then $m = ([\pi] + (1-[\pi]))m$

$= [\pi]m + (1-[\pi])m$. So $M = [\pi]M + (1-[\pi])M$.

If $m \in [\pi]M \cap (1-[\pi])M$ then $m = [\pi](1-[\pi])m = ([\pi]-[\pi])m = 0$.

The regular representation

The regular representation of A is the vector space A with A-action given by left multiplication

$$\begin{array}{rl} \rho_A: A & \to End(A) \\ a & \mapsto a_A \end{array} \quad \text{is injective}$$

because $a \cdot 1 = a$ implies $\ker \rho_A = 0$.

Identify A with $\operatorname{im}\rho_A$, so that A "is" a set of matrices. A matrix a is nilpotent if

$$a^k = 0, \text{ for some } k \in \mathbb{Z}_{>0}.$$

Proposition Let $\begin{array}{rl} t: A & \to \mathbb{C} \\ a & \mapsto Tr(a_A) \end{array}$ be the trace of the regular representation. Then

$Rad(\langle\ \rangle)$ is the largest ideal of A such that every element is nilpotent.

Proof To show: (a) Every element of $Rad(\langle\ \rangle)$ is nilpotent

(b) If J is an ideal of A and every element of J is nilpotent then $J \subseteq Rad(\langle\ \rangle)$.

(a) Let $r \in Rad(\langle\ \rangle)$. Then

$$Tr(r^k) = \langle r^k, 1\rangle = \langle r, r^{k-1}\rangle = 0, \text{ for all } k \in \mathbb{Z}_{>0}.$$

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of r. Then

$$p_k(\lambda_1, \ldots, \lambda_n) = \lambda_1^k + \cdots + \lambda_n^k = Tr(r^k) = 0, \text{ for all } k \in \mathbb{Z}_{>0}.$$

Since $p_k(\lambda_1,\ldots,\lambda_n)=0$, for $k\in\mathbb{Z}_{>0}$, (*)

$$e_k(\lambda_1,\ldots,\lambda_n)=0 \text{ for } k\in\mathbb{Z}_{>0}.$$

So
$$\prod_{i=1}^{n}(z+\lambda_i) = \sum_{k=0}^{n} e_k(\lambda_1,\ldots,\lambda_n)z^{n-k} = z^n$$

So $\lambda_1=\cdots=\lambda_n=0$. Thus, by Jordan normal form, r is nilpotent (conjugate to a strictly upper triangular matrix).

(b) Assume J is an ideal of A and all elements of J are nilpotent.
Let $r\in J$ and $a\in A$. Then $ra\in J$ and, since ra is nilpotent,

$$0=Tr(ra)=\langle r,a\rangle. \quad \text{So } r\in Rad(\langle\ \rangle).$$ //.

Expansion of (*):
Since $p_k(\lambda_1,\ldots,\lambda_n)=0$ for $k\in\mathbb{Z}_{>0}$

$$1=e^{-\sum_{k\in\mathbb{Z}_{>0}}\frac{p_k(\lambda_1,\ldots,\lambda_n)}{k}(-z)^k} = e^{-\sum_{i=1}^{n}\sum_{k\in\mathbb{Z}_{>0}}\frac{(-\lambda_i z)^k}{k}}$$

$$=\prod_{i=1}^{n}e^{-\ln\left(\frac{1}{1+\lambda_i z}\right)} = \prod_{i=1}^{n}e^{\ln(1+\lambda_i z)} = \prod_{i=1}^{n}(1+\lambda_i z)$$

$$=\sum_{k\in\mathbb{Z}_{\geq 0}} e_k(\lambda_1,\ldots,\lambda_n)z^k,$$

where the 3rd equality follows from

$$\frac{1}{1-x}=1+x+x^2+\cdots, \text{ so that } \frac{1}{1+x}=1+(-x)+(-x)^2+\cdots$$

and

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$= -(-x) - \frac{(-x)^2}{2} - \frac{(-x)^3}{3} - \frac{(-x)^4}{4} - \cdots$$

$$= -\sum_{k \in \mathbb{Z}_{>0}} \frac{(-x)^k}{k}.$$