Lecture 23: Group theory and linear algebra 14.09.2011

Let G be a group and let H be a subgroup of G.

The <u>set of cosets of H in G</u> is

$$G/H = \{gH \mid g \in G\},$$

where $gH = \{gh \mid h \in H\}$ for $g \in G$.

A <u>normal subgroup of G</u> is a subgroup K of G such that

if $g \in G$ and $k \in K$ then $gkg^{-1} \in K$.

<u>Theorem</u> Let G be a group and let H be a subgroup of G. Then

$$G/H \text{ with } \begin{array}{ccc} G/H \times G/H & \to & G/H \\ (g_1H, g_2H) & \mapsto & g_1g_2H \end{array}$$ is a well defined group

if and only if

H is a normal subgroup of G.

<u>Example</u> If $G = \{111, X1, 1X, ⋇, ⋇, ⋇\} = S_3$ and
$H = \{111, X1\}$
the cosets in G/H are

$111 \cdot H = X1 \cdot H = \{111, X1\}$

$1X \cdot H = ⋇ \cdot H = \{1X, ⋇\}$

$⋇ \cdot H = ⋇ \cdot H = \{⋇, ⋇\}$

If we try ~~to~~ define

$$(*) \qquad \begin{aligned} G/H \times G/H &\longrightarrow G/H \\ (g_1H, g_2H) &\longmapsto g_1g_2H \end{aligned}$$

then
$$(\mathrm{1X}\cdot H)(\mathrm{1X}\cdot H) = \mathrm{111}\cdot H = \{\mathrm{111}, \mathrm{X1}\}$$

$$= (\mathrm{X\!\!\!X}\cdot H)(\mathrm{X\!\!\!X}\cdot H) = \mathrm{X\!\!\!X}\cdot H = \{\divideontimes, \mathrm{X\!\!\!X}\}$$

gives
$$\{\mathrm{1X}, \mathrm{X\!\!\!X}\}^2 = \{\mathrm{111}, \mathrm{X1}\} \quad \text{and} \quad \{\mathrm{1X}, \mathrm{X\!\!\!X}\}^2 = \{\divideontimes, \mathrm{X\!\!\!X}\}$$

In other words, for this H, $(*)$ is <u>not</u> a function!
The theorem tells us that this is "because" H is not normal: $\mathrm{X1} \in H$, but

$$\mathrm{1X}\cdot\mathrm{X1}\cdot(\mathrm{1X})^{-1} = \mathrm{1X}\cdot\mathrm{X1}\cdot\mathrm{1X} = \begin{matrix}\mathrm{1X}\\ \mathrm{X1}\\ \mathrm{1X}\end{matrix} = \divideontimes \text{ is } \underline{\text{not}} \text{ in } H.$$

<u>Proof of the theorem</u>

$\Leftarrow$: Assume H is a normal subgroup

To show: (a)
$$\begin{aligned} G/H \times G/H &\to G/H \\ (g_1H, g_2H) &\mapsto g_1g_2H \end{aligned} \quad \text{is a function.}$$

~~(b)~~

(b) Using $(g_1H)(g_2H) = g_1g_2H$ as product:

(ba) If $g_1H, g_2H, g_3H \in G/H$ then
$$(g_1H\, g_2H)\, g_3H = g_1H\,(g_2H\, g_3H)$$

(bb) There exists $eH \in G/H$ such that if $gH \in G/H$ then $(eH)(gH) = gH$ and $(gH)(eH) = gH$

(bc) If $gH \in G/H$ then there exists ~~[illegible]~~ $xH \in G/H$

such that $(xH)(gH) = eH$ and $(gH)(xH) = eH$. ③

(a) To show: ~~If $g_1H = g_1'H$ and $g_2H = g_2'H$~~

If $(g_1H, g_2H) = (g_1'H, g_2'H)$ then $g_1g_2H = g_1'g_2'H$

Assume $(g_1H, g_2H) = (g_1'H, g_2'H)$.

Then $g_1H = g_1'H$ and $g_2H = g_2'H$.

To show: $g_1g_2H = g_1'g_2'H$.

To show: $g_1g_2 \in g_1'g_2'H$, since the cosets partition G.

We know $g_1 \in g_1'H$ and $g_2 \in g_2'H$.

So there exist $h_1, h_2 \in H$ such that $g_1 = g_1'h_1$ and $g_2 = g_2'h_2$

To show: $g_1g_2 \in g_1'g_2'H$.

$$\begin{aligned} g_1g_2 &= g_1'h_1g_2'h_2 \\ &= g_1'g_2'(g_2')^{-1}h_1g_2'h_2 \\ &= g_1'g_2'((g_2')^{-1}h_1g_2')h_2 \in g_1'g_2'H \end{aligned}$$

since H is a normal subgroup of G.

(b) ~~To show: (ba)~~

Use, $(g_1H)(g_2H) = g_1g_2H$ as product in G/H.

(ba) To show: If $g_1H, g_2H, g_3H \in G/H$ then $(g_1Hg_2H)g_3H$

$= g_1H(g_2Hg_3H)$

Assume $g_1H, g_2H, g_3H \in G/H$.

To show: $(g_1H\, g_2H)\, g_3H = g_1H\,(g_2H\, g_3H)$

Then

$$(g_1H\, g_2H)\, g_3H = g_1g_2H \cdot g_3H = (g_1g_2)g_3H$$

and

$$g_1H(g_2H\, g_3H) = g_1H \cdot g_2g_3H = g_1(g_2g_3)H$$

and $(g_1g_2)g_3 = g_1(g_2g_3)$ since G is associative.

So $(g_1H\, g_2H)\, g_3H = g_1H\,(g_2H\, g_3H)$.

(bb) To show: There exist $eH \in G/H$ such that
if $gH \in G/H$ then $(eH)(gH) = gH$ and $(gH)(eH) = gH$.

Let $eH = H$.

To show: if $gH \in G/H$ then $(eH)(gH) = gH$ and $(gH)(eH) = gH$.

Assume $gH \in G/H$.

To show: (bba) $eH \cdot gH = gH$

(bbb) $gH \cdot eH = gH$

(bba) $eH \cdot gH = 1 \cdot H gH = (1g)H = gH$,
since 1 is an identity for G.

(bbb) $gH\, eH = gH\, 1 \cdot H = (g1) \cdot H = gH$.

(bc) To show: If $gH \in G/H$ then there exists $xH \in G/H$
such that $(gH)(xH) = eH$ and $xH\, gH = H$.

Assume $gH \in G/H$.

To show: There exists $xH \in G/H$ such that

$$gHxH = H \text{ and } xHgH = H.$$

Let $xH = g^{-1}H$.

To show (bca) $gHxH = H$

(bcb) $xHgH = H$.

(bca) $gHxH = gHg^{-1}H = gg^{-1}H = 1 \cdot H = H$.

(bcb) $xHgH = g^{-1}HgH = (gg^{-1})H = 1 \cdot H = H$.

This completes the proof that if H is a normal subgroup of G then

G/H with $(g_1H)(g_2H) = g_1g_2H$ is a group.