

Week 3 Problem Sheet

Group Theory and Linear algebra

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1. Week 3: Vocabulary

- (1) Define a vector space and give some illustrative examples.
- (2) Define subspace and the intersection and sum of subspaces and give some illustrative examples.
- (3) Define a similar matrices and give some illustrative examples.
- (4) Define the change of basis matrix and give some illustrative examples.
- (5) Define the kernel, image, rank and nullity of a linear transformation and give some illustrative examples.
- (6) Define a linear transformation and give some illustrative examples.
- (7) Define basis and dimension and give some illustrative examples.
- (8) Define linearly dependent and linearly independent vectors and give some illustrative examples.
- (9) Define linear combination, linearly dependent and linearly independent and give some illustrative examples.

2. Week 3: Results

- (1) Show that any subset of a linearly independent set is also linearly independent.
- (2) Let \mathbb{F} be a field and let $E_{ij} \in M_{m \times n}(\mathbb{F})$ be the matrix with 1 in the i, j position and 0 elsewhere. Show that $\{E_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ is a basis of $M_{m \times n}(\mathbb{F})$.

- (3) Let $m, n \in \mathbb{Z}_{>0}$. Define $M_{m \times n}(\mathbb{R})$, addition and scalar multiplication, and show that $M_{m \times n}(\mathbb{R})$ is a vector space.
- (4) Let \mathbb{F} be a field. Define $\mathbb{F}[t]$, addition and scalar multiplication, and show that $\mathbb{F}[t]$ is a vector space.
- (5) Let S be a set and let \mathbb{F} be a field. Define addition and scalar multiplication on $\mathcal{F}(S, \mathbb{F}) = \{f: S \rightarrow \mathbb{F}\}$, the set of functions from S to \mathbb{F} , and show that $\mathcal{F}(S, \mathbb{F})$ is a vector space.
- (6) Let B_V and B'_V be bases of V and let P be the change of basis matrix from B_V and B'_V . Let B_W and B'_W be bases of W and let Q be the change of basis matrix from B_W and B'_W . Let $f: V \rightarrow W$ be a linear transformation and let A be the matrix of f with respect to the bases B_V and B_W .
- Show that P and Q are invertible.
 - Show that the matrix of f with respect to the bases B'_V and B'_W is QAP^{-1} .
- (7) Let $f: V \rightarrow W$ be a linear transformation.
- Show that the nullspace of f is a subspace of V .
 - Show that the image of f is a subspace of W .
- (8) Let $f: V \rightarrow W$ be a linear transformation and assume that V is finite dimensional. Show that the nullity of f plus the rank of f is equal to the dimension of V .
- (9) Let U and W be subspaces of a vector space V and assume that $U + W$ is finite dimensional. Then $\dim(U + W) + \dim(U \cap W) = \dim(U) + \dim(W)$.
- (10) Show that every vector space has a basis. In fact, every spanning set contains a basis and every linearly independent set can be extended to a basis.
- (11) Show that if B_1 and B_2 are two bases of a vector space then they have the same number of elements. (This means that you need to show that there exists a bijective function $f: B_1 \rightarrow B_2$.)
- (12) Show that a subset S of a vector space V is linearly dependent if and only if, there exists $s \in S$ which is a linear combination of the others.
- (13) If S is a non-empty subset of V , then $\text{span}(S)$ is a subspace of V .
- (14) Let V be a vector space over \mathbb{F} . A subset W of V is a subspace if and only if the following three conditions are satisfied:

- (1) W is non-empty,
 - (2) if $u, w \in W$ then $u + w \in W$,
 - (3) if $a \in \mathbb{F}$ and $w \in W$ then $aw \in W$.
- (15) Let $f: V \rightarrow V$ be a linear transformation on a finite dimensional vector space V . Show that the nullity of f is zero if and only if f is surjective.
 - (16) Let V be a vector space. Show that if U and W are subspaces of V then $U + W = \{u + w \mid u \in U \text{ and } w \in W\}$ is a subspace of V .
 - (17) Let V be a vector space. Show that if U and W are subspaces of V then $U \cap W$ is a subspace of V .
 - (18) Let V be a vector space. Show that if U and W are subspaces of V and $U \cup W = V$ then $U = V$ or $W = V$.

3. Week 3: Examples and Computations

- (1) Define \mathbb{R}^3 , addition and scalar multiplication, and show that \mathbb{R}^3 is a vector space.
- (2) Let \mathbb{F} be a field and $n \in \mathbb{Z}_{>0}$. Define \mathbb{F}^n , addition and scalar multiplication, and show that \mathbb{F}^n is a vector space.
- (3) Let \mathbb{F} be a field. Define $\mathcal{P}_n(\mathbb{F}) = \{a_0 + a_1t + \dots + a_nt^n \mid a_0, a_1, \dots, a_n \in \mathbb{F}\}$, addition and scalar multiplication, and show that $\mathcal{P}_n(\mathbb{F})$ is a vector space.
- (4) Define addition and scalar multiplication on $\mathcal{F}(\mathbb{R}, \mathbb{R}) = \{f: \mathbb{R} \rightarrow \mathbb{R}\}$, the set of functions from \mathbb{R} to \mathbb{R} , and show that $\mathcal{F}(\mathbb{R}, \mathbb{R})$ is a vector space.
- (5) Define addition and scalar multiplication on the set \mathcal{S} of solutions y of the differential equation

$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 23y = 0$$
 and show that \mathcal{S} is a vector space.
- (6) Define addition and scalar multiplication on the set $\ell^\infty = \{(a_1, a_2, \dots) \mid a_i \in \mathbb{R} \text{ for } i = 1, 2, \dots\}$ and show that ℓ^∞ is a vector space.
- (7) Define addition and scalar multiplication on the set $c_0 = \{(a_1, a_2, \dots) \mid a_i \in \mathbb{R} \text{ and } \lim_{i \rightarrow \infty} a_i = 0\}$ and show that c_0 is a vector space.

- (8) Show that $\{(a_1, a_2, \dots) \mid a_i \in \mathbb{R} \text{ and } \lim_{i \rightarrow \infty} a_i = 1\}$ is not a subspace of ℓ^∞ is a vector space.
- (9) Show that $W = \{(a, b, c) \mid a, b, c \in \mathbb{R} \text{ and } a + b + c = 0\}$ is a subspace of \mathbb{R}^3 .
- (10) Show that the set of matrices of trace zero is a subspace of the vector space $M_n(\mathbb{R})$.
- (11) Show that the set of polynomials with zero constant term is a subspace of the vector space $\mathbb{R}[t]$.
- (12) Show that the set of differentiable functions is a subspace of the vector space $\mathcal{F}(\mathbb{R}, \mathbb{R})$ of functions from \mathbb{R} to \mathbb{R} .
- (13) Show that the set c_0 of sequences such that $\lim_{i \rightarrow \infty} a_i = 0$ is a subspace of the vector space of sequences ℓ^∞ .
- (14) Show that the set of linear combinations of the vectors $(1, -2, 3)$ and $(0, 2, 1)$ in \mathbb{R}^3 is the set $\{(a, -2a + 2b, 3 + b) \mid a, b \in \mathbb{R}\}$.
- (15) Show that the set of linear combinations of the matrices
- $$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
- in $M_3(\mathbb{R})$ is the set of matrices of the form
- $$\begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \quad \text{where } a, b, c \in \mathbb{R}$$
- .
- (16) Show that the set $\{(1, 2, 3), (2, -1, 0), (-1, 8, 9)\}$ is linearly dependent in \mathbb{R}^3 .
- (17) Show that the set $\{1, x, x^2, 1 + x^3\}$ is linearly independent in $\mathbb{R}[x]$.
- (18) Show that the set $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & -29 \\ 0 & 0 \end{pmatrix} \right\}$ is linearly dependent in $M_2(\mathbb{R})$.
- (19) Let \mathbb{F} be a field and let $n \in \mathbb{Z}_{>0}$. Show that $\{e_1 = (1, 0, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0), \dots, e_n = (0, 0, 0, \dots, 1)\}$ is a basis of \mathbb{F}^n .
- (20) Show that the set $\{(2, 1, 3), (1, 2, 3), (1, 0, 0)\}$ is a basis of \mathbb{R}^3 .

- (21) Show that the set $\{1, x, x^2, 1 + x^3\}$ is a basis of the vector space of polynomials with coefficients in \mathbb{R} of degree ≤ 3 .
- (22) Show that the set $\{1, x, x^2, x^3, \dots\}$ is a basis of the vector space $\mathbb{R}[x]$.
- (23) Show that \mathbb{R}^3 has dimension 3.
- (24) Let \mathbb{F} be a field and let $n \in \mathbb{Z}_{>0}$. Show that \mathbb{F}^n has dimension n .
- (25) Let $m, n \in \mathbb{Z}_{>0}$. Show that $M_{m \times n}(\mathbb{R})$ has dimension mn .
- (26) Show that the set of polynomials with coefficients in \mathbb{R} and degree $\leq n$ has dimension $n + 1$.
- (27) Show that the vector space \mathcal{S} of solutions y of the differential equation
- $$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 23y = 0$$
- has dimension 2.
- (28) Show that ℓ^∞ has infinite dimension.
- (29) Show that c_0 has infinite dimension.
- (30) Let \mathbb{F} be a field. Show that $\mathbb{F}[t]$ has infinite dimension.
- (31) Show that rotation about the origin through a fixed angle θ is a linear transformation on \mathbb{R}^2 .
- (32) Show that rotation about any line through and through a fixed angle θ is a linear transformation on \mathbb{R}^3 .
- (33) Show that differentiation with respect to t is a linear transformation on $\mathbb{R}[t]$.
- (34) Let $C(\mathbb{R}) = \{f: \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ is continuous}\}$, a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$. Let $I: C(\mathbb{R}) \rightarrow C(\mathbb{R})$ be given by
- $$I(f)(t) = \int_0^t f(x) dx.$$
- Show that I is a linear transformation.
- (35) Show that the functions $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^2$ and $g(x) = x + 2$ are not linear transformations.
- (36) Show that rotation in \mathbb{R}^2 has kernel $\{0\}$ and image \mathbb{R}^2 .
- (37) Show that differentiation with respect to x on $\mathbb{R}[x]$ has kernel $\mathbb{R} \cdot 1$ and image $\mathbb{R}[x]$.
- (38) Rotation about the origin through a fixed angle θ is a linear transformation f on \mathbb{R}^2 .

- Find the matrix of f with respect to the basis $\{(1, 0), (0, 1)\}$.
- (39) Differentiation with respect to t is a linear transformation f on $\mathbb{R}[t]$. Find the matrix of f with respect to the basis $\{1, t, t^2, \dots\}$.
- (40) Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation given by $f(x, y) = (3x - y, -x + 3y)$. Let $\mathcal{B} = \{(1, 0), (0, 1)\}$ and let $\mathcal{C} = \{(1, 1), (-1, 1)\}$. Find the change of basis matrix P from \mathcal{B} to \mathcal{C} and the change of basis matrix Q from \mathcal{C} to \mathcal{B} . Find the matrix A of f with respect to the basis \mathcal{B} and the matrix B of f with respect to the basis \mathcal{C} . Verify that $A = PBQ$.
- (41) In the vector space $(\mathbb{Z}/7\mathbb{Z})^4$ determine whether the set $\{(1, 3, 0, 2), (2, 1, 3, 0)\}$ is linearly dependent and whether it is a basis.
- (42) In the vector space $(\mathbb{Z}/7\mathbb{Z})^4$ determine whether the set $\{(1, 2, 3, 1), (4, 6, 2, 0), (0, 1, 5, 1)\}$ is linearly dependent and whether it is a basis.
- (43) In the vector space $(\mathbb{Z}/7\mathbb{Z})^4$ determine whether the set $\{(1, 2, 3, 1), (4, 6, 2, 0), (0, 1, 5, 2), (0, 1, 1, 0), (0, 1, 0, 1)\}$ is linearly dependent and whether it is a basis.
- (44) In the vector space $M_2(\mathbb{R})$ determine whether the set
- $$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$
- is linearly dependent and whether it is a basis.
- (45) In the vector space $M_2(\mathbb{R})$ determine whether the set
- $$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$
- is linearly dependent and whether it is a basis.
- (46) In the vector space $M_2(\mathbb{R})$ determine whether the set
- $$\left\{ \begin{pmatrix} 2 & 0 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 4 & -6 \\ 3 & 8 \end{pmatrix} \right\}$$
- is linearly dependent and whether it is a basis.
- (47) What is the dimension of the space $M_3(\mathbb{Z}/5\mathbb{Z})$?
- (48) Let $B = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}$. Show that the function $g: M_2(\mathbb{R}) \rightarrow M_2(\mathbb{R})$ given by $g(A) = AB$, is a linear transformation.

(49) Let $B = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}$. Find the matrix of the linear transformation $g: M_2(\mathbb{R}) \rightarrow M_2(\mathbb{R})$ given

by $g(A) = AB$, with respect to the basis

$$\left\{ E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

(50) Find the matrix, with respect to the standard basis of \mathbb{R}^2 , of the reflection in the x -axis. Let $a, b, c, d \in \mathbb{R}$ such that $ad - bc \neq 0$. Let B be the basis of \mathbb{R}^2 given by $\{(a, b), (c, d)\}$. Determine the change of basis matrix from the standard basis of \mathbb{R}^2 to B and use it to calculate the matrix of the reflection with respect to the basis B .

(51) Calculate the nullity and rank of the linear transformation f on \mathbb{R}^3 given by

$$f(e_1) = e_1 - e_2, \quad f(e_2) = e_2 - e_3, \quad f(e_3) = e_1 - e_3,$$

where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$.

(52) Calculate the nullity and rank of the linear transformation f on $(\mathbb{Z}/7\mathbb{Z})^3$ given by

$$f(1, 0, 0) = (1, 2, 3), \quad f(0, 1, 0) = (3, 4, 5), \quad f(0, 0, 1) = (5, 1, 4).$$

(53) Determine whether the set of upper triangular matrices with real entries

$$\{A = (a_{ij}) \in M_3(\mathbb{R}) \mid a_{ij} = 0 \text{ for } i > j\},$$

is a vector space over \mathbb{R} .

(54) Determine whether the set of functions $f: \mathbb{R} \rightarrow \mathbb{R}$ such that $f(0) \geq 0$ is a vector space over \mathbb{R} .

(55) Consider the subset $S = \{(1, 3), (3, 4), (2, 3)\}$, in $(\mathbb{Z}/5\mathbb{Z})^2$.

(i) Does S span $(\mathbb{Z}/5\mathbb{Z})^2$?

(ii) Is S linearly independent?

(iii) Find a subset of S which is a basis of $(\mathbb{Z}/5\mathbb{Z})^2$.

(56) Let U, W be 3-dimensional subspaces of \mathbb{R}^5 . Show that $U \cap W$ contains a non-zero vector.

(57) Define $f: M_3(\mathbb{R}) \rightarrow M_3(\mathbb{R})$ by $f(A) = A + A^t$, where A^t is the transpose of A .

(i) Show that f is a linear transformation.

(ii) Describe the kernel and image of f .

(iii) Find bases for these spaces, and verify that the rank-nullity formula holds.

- (58) Are the following sets of functions from \mathbb{R} to \mathbb{R} linearly independent?
- (i) $\{1, \sin^2 x, \cos^2 x\}$,
 - (ii) $\{1, \sin(2x), \cos(2x)\}$.
- (59) Show that $\{1, \sqrt{2}, \sqrt{3}\}$ is linearly independent over the field \mathbb{Q} .
- (60) Let $\beta = \sqrt[3]{2}$. Then $V = \{x + y\beta + z\beta^2 \mid x, y, z \in \mathbb{Q}\}$ is a vector space over the field \mathbb{Q} .
- (a) Show that V is closed under multiplication.
 - (b) Let α be a nonzero element of V , and let $f: V \rightarrow V$ be multiplication by α , $f(v) = \alpha v$. Show that f is a linear transformation and determine the kernel and the image of f .
 - (c) Show that V is a field.

4. References

[GH] [J.R.J. Groves](#) and [C.D. Hodgson](#), *Notes for 620-297: Group Theory and Linear Algebra*, 2009.

[Ra] [A. Ram](#), *Notes in abstract algebra*, University of Wisconsin, Madison 1994.