Problem Set -- Sets, Orders and functions
620-295 Semester I 2010

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Last updates: 2 May 2010

(1) Absolute value and inequalities
(2) Induction, or perhaps not
(3) Orders on $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$
(4) Cardinality
(5) Sets and functions

1. Absolute value and inequalities

(1) Let $x \in \mathbb{R}$. Define $|x|$.

(2) Let $x \in \mathbb{C}$. Define $|x|$.

(3) Let $x \in \mathbb{R}^n$. Define $|x|$.

(4) Let $x \in \mathbb{R}$. Show that $|x| = |x + 0i|$.

(5) State and prove Lagrange's identity for $\mathbb{R}$.

(6) State and prove the Schwarz identity for $\mathbb{R}$.

(7) State and prove Lagrange's identity for $\mathbb{R}^2$.

(8) State and prove the Schwarz identity for $\mathbb{R}^2$.

(9) Let $x \in \mathbb{R}^n$. Show that $|-x| = |x|$.

(10) Let $x, y \in \mathbb{R}$. Show that $|x + y| \leq |x| + |y|$.

(11) Let $x, y \in \mathbb{C}$. Show that $|x + y| \leq |x| + |y|$.

(12) Let $x,\ y \in \mathbb{R}^n$. Show that $|x + y| \leq |x| + |y|$.

(13) Let $x,\ y,\ z \in \mathbb{R}^n$. Show that $|x + y + z| \leq |x| + |y| + |z|$.

(14) Let $x,\ y \in \mathbb{C}$. Show that $|x + y|^2 + |x - y|^2 = 2(|x|^2 + |y|^2)$. Is this identity true for $x,\ y \in \mathbb{R}^n$?

(15) Let $x,\ y \in \mathbb{C}$. Show that $|x + y|^2 = |x|^2 + |y|^2 + 2\mathrm{Re}(x\bar{y})$.

(16) Let $x,\ y \in \mathbb{R}$. Show that $|x + y| \geq |\ |x| - |y|\ |$.

(17) Let $x,\ y \in \mathbb{R}$. Show that $|x - y| \geq |\ |x| - |y|\ |$.

(18) Let $x,\ y,\ z \in \mathbb{R}$. Show that $|x + y + z| \geq |\ |x| - |y| - |z|\ |$.

(19) For $x \in \mathbb{R}$, give solutions to the following inequalities in terms of intervals:
(a) $|x| > 3$.
(b) $|1 + 2x| \geq 4$.
(c) $|x + 2| \geq 5$.

(20) For $x \in \mathbb{R}$, rewrite each of the following inequalities in terms of intervals:
(a) $|x + 3| > 1$
(b) $|x - 2| < 3$
(c) $|x + 2| \leq 2$ and $|x| > 1$
(d) $|x + 2| \leq 2$ or $|x| > 1$

(21) For $x \in \mathbb{R}$, give solutions to the following inequalities in terms of intervals:
(a) $|x - 2| < 3$ or $|x + 1| < 1$.
(b) $|x - 2| < 3$ and $|x + 1| < 1$.
(c) $|x - 5| < |x + 1|$.

(22) Let $a,\ b \in \mathbb{R}$ and let $\varepsilon \in \mathbb{R}$ such that $0 < \varepsilon < |b|$. Show that $\left|\dfrac{a + \varepsilon}{b + \varepsilon}\right| \leq \dfrac{|a| + \varepsilon}{|b| + \varepsilon}$.

(23) Prove that if $a_1,\ a_2,\ \ldots,\ a_n \in \mathbb{R}$ then $\left|\sum_{k=1}^{n} a_k\right| \leq \sum_{k=1}^{n} |a_k|$. Is this identity true for $a_1,\ a_2,\ \ldots,\ a_n \in \mathbb{R}^n$?

(24) Prove that if $a_1,\ a_2,\ \ldots,\ a_n \in \mathbb{R}$ then $\left|\sum_{k=1}^{n} a_k\right| \leq |a_p| - \sum_{k=1, k\neq p}^{n} |a_k|$. Is this identity true for $a_1,\ a_2,\ \ldots,\ a_n \in \mathbb{R}^n$?

(25) Find the minimal $N \in \mathbb{Z}_{>0}$ such that $n < 2^n$ for all $n \geq N$.

(26) Find the minimal $N \in \mathbb{Z}_{>0}$ such that $n! > 2^n$ for all $n \geq N$.

(27) Find the minimal $N \in \mathbb{Z}_{>0}$ such that $2^n > 2n^3$ for all $n \geq N$.

(28) (Bernoulli's inequality) Prove that if $a \in \mathbb{R}$ and $a >$ -1 then $(1+a)^n \geq 1 + na$ for $n \in \mathbb{Z}_{>0}$.

(29) Prove that if $x \in \mathbb{R}$ then $1 + x \leq e^x$.

(30) Prove that if $x \in \mathbb{R}_{>0}$ then $\log x \geq \dfrac{x-1}{x}$.

(31) Prove that if $x, y \in \mathbb{R}_{\geq 0}$ and $p \in \mathbb{R}$ with $0 < p < 1$ then $(x+y)^p \leq x^p + y^p$.

(32) (Jensen's inequality) Let I be an interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be a convex function. If $x_1, \ldots, x_n \in \mathbb{R}$ and $t_1, \ldots, t_n \in [0, 1]$ with $t_1 + \cdots + t_n = 1$, then $f(t_1 x_1 + \cdots + t_n x_n) \leq t_1 f(x_1) + \cdots + t_n f(x_n)$.

(33) If $x_1, \ldots, x_n \in \mathbb{R}_{\geq 0}$ and $t_1, \ldots, t_n \in \mathbb{R}_{\geq 0}$ with $t_1 + \cdots + t_n = 1$, then $t_1 x_1 + \cdots + t_n x_n \geq {x_1}^{t_1} \cdots {x_n}^{t_n}$.

2. Induction, or perhaps not

(1) Prove that if $n \in \mathbb{Z}_{>0}$ then 3 is a factor of $n^3 - n + 3$.

(2) Prove that if $n \in \mathbb{Z}_{>0}$ then 9 is a factor of $10^{n+1} + 3 \cdot 10^n + 5$.

(3) Prove that if $n \in \mathbb{Z}_{>0}$ then 4 is a factor of $5^n - 1$.

(4) Prove that if $n \in \mathbb{Z}_{>0}$ then $x - y$ is a factor of $x^n - y^n$.

(5) Prove that if $n \in \mathbb{Z}_{>0}$ then $7^{2n} - 48n - 1$ is divisible by 2304.

(6) Prove that if $n \in \mathbb{Z}_{>0}$ then $2 + 4 + 6 + \cdots + 2n = n(n+1)$.

(7) Prove that if $n \in \mathbb{Z}_{>0}$ then $1 + 4 + 7 + \cdots + (3n-2) = \frac{1}{2} n(3n-1)$.

(8) Prove that if $n \in \mathbb{Z}_{>0}$ then $2 + 7 + 12 + \cdots + (5n-3) = \frac{1}{2} n(5n-1)$.

(9) Prove that if $n \in \mathbb{Z}_{>0}$ then $1 + 2 \cdot 2 + 3 \cdot 2^2 + 4 \cdot 2^3 + \cdots + n2^{n-1} = 1 + (n-1)2^n$.

(10) Prove that if $n \in \mathbb{Z}_{>0}$ then $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6} n(n+1)(2n+1)$.

(11) Prove that if $n \in \mathbb{Z}_{>0}$ then $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\cdots+\dfrac{1}{n\cdot(n+1)}=\dfrac{n}{n+1}$.

(12) Prove that if $n \in \mathbb{Z}_{>0}$ then $3+3^2+3^3+\cdots+3^n=\frac{3}{2}(3^n-1)$.

(13) Prove that if $n \in \mathbb{Z}_{>0}$ then $(1+2^5+\cdots+n^5)+(1+2^7+\cdots+n^7)=2\left(\frac{n(n+1)}{2}\right)^4$.

(14) Prove that if $n \in \mathbb{Z}_{>0}$ then $1+r+r^2+\cdots+r^n=\dfrac{1-r^n}{1-r}$.

(15) Prove that if $n \in \mathbb{Z}_{>0}$ then $\sum\limits_{k=1}^{n}(2k-1)=n^2$.

(16) Prove that $\sum\limits_{k=1}^{n} k=\frac{1}{2}n(n+1)$.

(17) Prove that $\sum\limits_{k=1}^{n}(3k-2)=\frac{1}{2}n(3n-1)$.

(18) Prove that $\sum\limits_{k=1}^{n} k^2=\frac{1}{6}n(n+1)(2n+1)$.

(19) Prove that $\sum\limits_{k=1}^{n} k^3=\frac{1}{4}n^2(n+1)^2$.

(20) Prove that $\sum\limits_{k=1}^{n} k^3=\left(\sum\limits_{k=1}^{n} k\right)^2$.

(21) Prove that $\sum\limits_{k=1}^{n}\dfrac{1}{k(k+1)}=\dfrac{n}{n+1}$.

(22) Define a sequence by $a_1=0$, $a_{2k}=\frac{1}{2}a_{2k-1}$ and $a_{2k+1}=\frac{1}{2}+a_{2k}$. Show that $a_{2k}=\frac{1}{2}-\left(\frac{1}{2}\right)^{\mathrm{k}}$.

(23) Prove that if $n \in \mathbb{Z}_{>0}$ then 3 is a factor of n^3-n+3.

(24) Prove that if $n \in \mathbb{Z}_{>0}$ then 9 is a factor of $10^{n+1}+3\cdot 10^n+5$.

(25) Prove that if $n \in \mathbb{Z}_{>0}$ then 4 is a factor of 5^n-1.

(26) Prove that if $n \in \mathbb{Z}_{>0}$ then $x-y$ is a factor of x^n-y^n.

(27) Let D be a diagonal matrix, $D=\operatorname{diag}(\lambda_1,\ \ldots\ \lambda_s)$, where $D_{ii}=\lambda_i$, $D_{ij}=0$, for $i \neq j$.

Prove, by induction that, for each positive integer n,
$$D^n = \mathrm{diag}(\lambda_1^n, \ldots, \lambda_s^n).$$

(28) Let A be a matrix such that $A = PDP^{-1}$, where D is diagonal. Prove, by induction, that for each positive integer n,
$$A^n = PD^nP^{-1}.$$

3. Orders on $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$

(1) Define the order $\geq$ on $\mathbb{Z}_{>0}$.

(2) Define the order $\geq$ on $\mathbb{Z}_{\geq 0}$.

(3) Define the order $\geq$ on $\mathbb{Z}$.

(4) Define the order $\geq$ on $\mathbb{Q}$.

(5) Show that $\frac{a}{b} \leq \frac{c}{d}$ if and only if $abd^2 \leq cdb^2$.

(6) Define the order $\geq$ on $\mathbb{R}$.

(7) Show that there is no order $\geq$ on $\mathbb{C}$ such that $\mathbb{C}$ is a totally ordered field.

(8) Show that if $x, y, z \in \mathbb{R}$ and $x \leq y$ and $y \leq z$ then $x \leq z$.

(9) Show that if $x, y \in \mathbb{R}$ and $x \leq y$ and $y \leq x$ then $x = y$.

(10) Show that if $x, y, z \in \mathbb{R}$ and $x \leq y$ then $x + z \leq y + z$.

(11) Show that if $x, y \in \mathbb{R}$ and $x \geq 0$ and $y \geq 0$ then $xy \geq 0$.

(12) Show that if $x \in \mathbb{R}$ and $x \neq 0$ then $x^2 > 0$.

(13) Show that if $x, y \in \mathbb{R}$ and $0 < x < y$ then $y^{-1} < x^{-1}$.

(14) (The Archimedean property of $\mathbb{R}$) Show that if $x, y \in \mathbb{R}$ and $x \in \mathbb{R}_{>0}$ then there exists $n \in \mathbb{Z}_{\geq 0}$ such that $nx > y$.

(15) Show that the Archimedean property is equivalent to $\mathbb{Z}_{>0}$ is an unbounded subset of $\mathbb{R}$.

(16) ($\mathbb{Q}$ is dense in $\mathbb{R}$) Show that if $x, y \in \mathbb{R}$ and $x < y$ then there exists $p \in \mathbb{Q}$ such that $x < p < y$.

(17) ($\mathbb{R} - \mathbb{Q}$ is dense $\mathbb{R}$) Show that if $x, y \in \mathbb{R}$ and $x < y$ then there exists $p \in \mathbb{R} - \mathbb{Q}$ such

that $x < p < y$.

(18) If $x,\ y \in \mathbb{R}$ and $x < y$ show that there exist infinitely many rational numbers between x and y as well as infinitely many irrational numbers.

(19) Let $x \in \mathbb{R}_{>0}$ and $n \in \mathbb{Z}_{>0}$. Then there exists a unique $y \in \mathbb{R}_{>0}$ such that $y^n = x$.

(20) For each of the following subsets of $\mathbb{R}$ find the maximum, the minimum, an upper bound, a lower bound, the supremum, and the infimum:

(a) $A = \{p \in \mathbb{Q} \mid p^2 < 2\}$,
(b) $B = \{p \in \mathbb{Q} \mid p^2 > 2\}$,
(c) $E_1 = \{r \in \mathbb{Q} \mid r < 0\}$,
(d) $E_2 = \{r \in \mathbb{Q} \mid r \leq 0\}$,

(21) For each of the following subsets of $\mathbb{R}$ find the maximum, the minimum, an upper bound, a lower bound, the supremum, and the infimum:

(a) $S = \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{Z}_{>0} \right\}$,
(b) $S = [0,\ 1)$,
(c) $S = \mathbb{Z}_{>0}$,
(d) $S = \{x \in \mathbb{Q} \mid x \leq 0 \quad \text{or} \quad (x > 0 \quad \text{and} \quad x^2 > 2)\}$,

(22) For each of the following subsets of $\mathbb{R}$ find the maximum, the minimum, an upper bound, a lower bound, the supremum, and the infimum:

(a) $S = \mathbb{Z}$,
(b) $S = [\sqrt{2},\ 2]$,
(c) $S = (\sqrt{2},\ 2)$,
(d) $S = \left\{ x \in \mathbb{R} \,\middle|\, x = \frac{(-1)^n}{n}, \quad n \in \mathbb{Z}_{>0} \right\}$,

(23) For each of the following subsets of $\mathbb{R}$ find the maximum, the minimum, an upper bound, a lower bound, the supremum, and the infimum:

(a) $S = \left\{ \frac{1}{(|n| + 1)^2} \,\middle|\, n \in \mathbb{Z} \right\}$,
(b) $S = \left\{ n + \frac{1}{n} \,\middle|\, n \in \mathbb{Z}_{>0} \right\}$,
(c) $S = \{2^{-m} - 3^n \mid m,\ n \in \mathbb{Z}_{\geq 0}\}$,

(d) $S = \{x \in \mathbb{R} \mid x^3 - 4x < 0\}$,

(24) For each of the following subsets of $\mathbb{R}$ find the maximum, the minimum, an upper bound, a lower bound, the supremum, and the infimum:

(a) $S = \{1 + x^2 \mid x \in \mathbb{R}\}$,
(b) $S = \{x \in \mathbb{R} \mid x^2 < 9\}$,
(c) $S = \{x \in \mathbb{R} \mid x^2 \leq 7\}$,
(d) $S = \{x \in \mathbb{R} \mid |x + 2| \leq 2 \quad \text{or} \quad |x| > 1\}$.

Are the supremum and infimum (if they exist) in the set S?

(25) For each of the following subsets of $\mathbb{R}$ find the maximum, the minimum, an upper bound, a lower bound, the supremum, and the infimum:

(a) $S = \{x \in \mathbb{R} \mid |2x + 1| < 5\}$,
(b) $S = \{x \in \mathbb{R} \mid |x - 2| < 3 \text{ and } |x + 1| < 1\}$,
(c) $S = \{x \in \mathbb{R} \mid x \in \mathbb{Q} \text{ and } x^2 < 7\}$,
(d) $S = \{x \in \mathbb{R} \mid |x + 2| \leq 2 \text{ or } |x| > 1\}$.

Are the supremum and infimum (if they exist) in the set S?

(26) Find an upper bound for the function $f(x) = \dfrac{2x^2 + 1}{x + 3}$ for $x \in \mathbb{R}$ and $|x| < 1$.

(27) Find an upper bound for the function $f(x) = \dfrac{x^3 + 3x + 1}{10 - x^2}$ for $x \in \mathbb{R}$ and $|x + 1| < 2$.

(28) Let S be a nonempty subset of $\mathbb{R}$. Show that $x = \sup S$ if and only if

(a) x is an upper bound of S, and
(b) for every $\varepsilon \in \mathbb{R}_{>0}$ there exists $y \in S$ such that $x - \varepsilon < y \leq x$.

(29) State and prove a characterization of $\inf S$ analogous to the characterization of $\sup S$ in the previous problem.

(30) Let $c \in \mathbb{R}$ and let S be a subset of $\mathbb{R}$. Show that if S is bounded then $c + S = \{c + s \mid s \in \mathbb{R}\}$ is bounded.

(31) Let $c \in \mathbb{R}$ and let S be a subset of $\mathbb{R}$. Show that if S is bounded then $cS = \{cs \mid s \in \mathbb{R}\}$ is bounded.

(32) Let $c \in \mathbb{R}$ and let S be a subset of $\mathbb{R}$. Show that $\sup(c + S) = c + \sup S$.

(33) Let $c \in \mathbb{R}_{\geq 0}$ and let S be a subset of $\mathbb{R}$. Show that $\sup(cS) = c \sup S$.

(34) Let $c \in \mathbb{R}$ and let S be a subset of $\mathbb{R}$. Show that $\inf(c + S) = c + \inf S$.

(35) Let $c \in \mathbb{R}_{\leq 0}$ and let S be a subset of $\mathbb{R}$. Show that $\inf(cS) = c \inf S$.

4. Cardinality

(1) Define (a) cardinality, (b) finite, (c) infinite, (d) countable, and (e) uncountable.

(2) Prove that $\mathrm{Card}(\{a, b, c, d, e\}) = \mathrm{Card}(\{1, 2, 3, 4, 5\})$.

(3) Show that $\mathrm{Card}(\mathbb{Z}_{>0}) = \mathrm{Card}(\mathbb{Z}_{\geq 0})$.

(4) Show that $\mathrm{Card}(\mathbb{Z}) = \mathrm{Card}(\mathbb{Z}_{\geq 0})$.

(5) Show that $\mathrm{Card}(\mathbb{Z}_{>0}) = \mathrm{Card}(\mathbb{Z})$.

(6) Show that $\mathrm{Card}(\{x \in \mathbb{Q} \mid 0 < x \leq 1\}) = \mathrm{Card}(\mathbb{Z}_{>0})$.

(7) Show that $\mathrm{Card}(\{x \in \mathbb{R} \mid 0 < x \leq 1\}) \neq \mathrm{Card}(\mathbb{Z}_{>0})$.

(8) Show that $\mathrm{Card}(\mathbb{Z}_{>0}) = \mathrm{Card}(\mathbb{Q})$.

(9) Show that $\mathrm{Card}(\mathbb{Z}_{>0}) \neq \mathrm{Card}(\mathbb{R})$.

(10) Show that $\mathrm{Card}(\mathbb{C}) = \mathrm{Card}(\mathbb{R})$.

(11) Let S be a set. Show that $\mathrm{Card}(S) = \mathrm{Card}(S)$.

(12) Show that if $\mathrm{Card}(S) = \mathrm{Card}(T)$ then $\mathrm{Card}(T) = \mathrm{Card}(S)$.

(13) Show that if $\mathrm{Card}(S) = \mathrm{Card}(T)$ and $\mathrm{Card}(T) = \mathrm{Card}(U)$ then $\mathrm{Card}(S) = \mathrm{Card}(U)$.

(14) Define $\mathrm{Card}(S) \leq \mathrm{Card}(T)$ if there exists an injective function $f : S \to T$. Show that if $\mathrm{Card}(S) \leq \mathrm{Card}(T)$ and $\mathrm{Card}(T) \leq \mathrm{Card}(S)$ then $\mathrm{Card}(S) = \mathrm{Card}(T)$.

5. Sets and functions

(1) Let A, B and C be sets. Show that $(A \cup B) \cup C = A \cup (B \cup C)$.

(2) Let A and B be sets. Show that $A \cup B = B \cup A$.

(3) Let A be a set. Show that $A \cup \emptyset = A$.

(4) Let A, B and C be sets. Show that $(A \cap B) \cap C = A \cap (B \cap C)$.

(5) Let A and B be sets. Show that $A \cap B = B \cap A$.

(6) Let A, B and C be sets. Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(7) Define (a) partial order, (b) total order, (c) partially ordered set, and (d) totally ordered set.

(8) Define (a) maximum, (b) minimum, (c) upper bound, (d) lower bound, (e) bounded above, (f) bounded below.

(9) Define (a) upper bound, (b) lower bound, (c) least upper bound, (d) greatest lower bound, (e) supremum and (f) infimum.

(10) Let S be a set. Show that the set of subsets of S is partially ordered by inclusion.

(11) Give an example of a partially ordered set S with more than one maximal element.

(12) Let S be a partially ordered set and let E be a subset of S. Show that if a greatest lower bound of E exists in S then it is unique.

(13) Show that $\mathbb{Q}$ does not have the least upper bound property.

(14) Show that $\mathbb{R}$ has the least upper bound property.

(15) Which of $\mathbb{Z}_{>0}$, $\mathbb{Z}_{\geq 0}$, $\mathbb{Z}$, $\mathbb{C}$ have the least upper bound property?

(16) Let S, T and U be sets and let $f : S \to T$ and $g : T \to U$ be functions. Show that

a. if f and g are injective then $g \circ f$ is injective,
b. if f and g are surjective then $g \circ f$ is surjective, and
c. if f and g are bijective then $g \circ f$ is bijective.

(17) Let $f : S \to T$ be a function and let $U \subseteq S$. The **image** of U under f is the subset of T given by

$$f(U) = \{f(u) \mid u \in U\}.$$

Let $f : S \to T$ be a function. The **image** of U under f is the subset of T given by

$$\operatorname{im} U = \{f(s) \mid s \in S\}.$$

Note that $\operatorname{im} f = f(S)$.

Let $f : S \to T$ be a function and let $V \subseteq T$. The **inverse image** of V under f is the subset of S given by

$$f^{-1}(V) = \{s \in S \mid f(s) \in V\}.$$

Let $f : S \to T$ be a function and let $t \in T$. The **fiber** of f over t is the subset of S given by

$$f^{-1}(t) = \{s \in S \mid f(s) = t\}.$$

Let $f : S \to T$ be a function. Show that the set $F = \left\{ f^{-1}(t) \mid t \in T \right\}$ of fibers of the map f is a partition of S.

(18) a. Let $f : S \to T$ be a function. Define

$$\begin{aligned} f' : S &\longrightarrow \operatorname{im} f \\ s &\longmapsto f(s). \end{aligned}$$

Show that the map f' is well defined and surjective.

b. Let $f : S \to T$ be a function and let $F = \left\{ f^{-1}(t) \mid t \in \operatorname{im} f \right\} = \left\{ f^{-1}(t) \mid t \in T \right\} \setminus \emptyset$ be the set of nonempty fibers of the map f. Define

$$\begin{aligned} \hat{f} : \quad F &\longrightarrow T \\ f^{-1}(t) &\longmapsto t \end{aligned}.$$

Show that the map $\hat{f}$ is well defined and injective.

c. Let $f : S \to T$ be a function and let $F = \left\{ f^{-1}(t) \mid t \in \operatorname{im} f \right\} = \left\{ f^{-1}(t) \mid t \in T \right\} \setminus \emptyset$ be the set of nonempty fibers of the map f. Define

$$\begin{aligned} \hat{f}' : \quad F &\longrightarrow \operatorname{im} T \\ f^{-1}(t) &\longmapsto t \end{aligned}.$$

Show that the map $\hat{f}'$ is well defined and bijective.

(19) Let be a set. The **power set** of S, 2^S, is the set of all subsets of S.

Let S be a set and let $\{0, 1\}^S$ be the set of all functions $f : S \to \{0, 1\}$. Given a subset $T \subseteq S$ define a function $f_T : S \to \{0, 1\}$ by

$$f_T(s) = \begin{cases} 0, & \text{if } s \notin T, \\ 1, & \text{if } s \in T. \end{cases}$$

Show that the map

$$\begin{aligned} \phi : \quad 2^S &\longrightarrow \{0, 1\}^S \\ T &\longmapsto f_T \end{aligned}$$

is a bijection.

(20) Let $\circ : S \times S \to S$ be an associative operation on a set S. An **identity** for $\circ$ is an element $e \in S$ such that $e \circ s = s \circ e = s$ for all $s \in S$.

Let e be an identity for an associative operation $\circ$ on a set S. Let $s \in S$. A **left inverse** for s is an element $t \in S$ such that $t \circ s = e$. A **right inverse** for s is an element $t' \in S$ such that $s \circ t' = e$. An **inverse** for s is an element $s^{-1} \in S$ such that $s^{-1} \circ s = s \circ s^{-1} = e$.

a. Let $\circ$ be an operation on a set S. Show that if S contains an identity for $\circ$ then it is unique.

b. Let e be an identity for an associative operation $\circ$ on a set S. Let $s \in S$. Show that if s has an inverse then it is then it is unique.

(21) a. Let S and T be sets and let ι_S and ι_T be the identity maps on S and T respectively. Show that for any function $f : S \to T$,

$$\iota_T \circ f = f, \qquad \text{and}$$

$$f \circ \iota_S = f.$$

b. Let $f : S \to T$ be a function. Show that if an inverse function to f exists then it is unique. (Hint: The proof is very similar to the proof in Ex. 5b above.)

6. References

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