Problem Set - Derivatives and Taylor approximations
620-295 Semester I 2010

Arun Ram
Department of Mathematics and Statistics
University of Melbourne
Parkville, VIC 3010 Australia
aram@unimelb.edu.au

and

Department of Mathematics
University of Wisconsin, Madison
Madison, WI 53706 USA
ram@math.wisc.edu

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1. Intermediate value property

(1) Find rigorous bounds on the location of all real zeros of $f(x) = x^7 - 27x^3 + 42$.

(2) Prove that $\sqrt{x}$ is continuous for $x \geq 0$.

(3) If $f(x) = x^3 - 5x^2 + 7x - 9$, prove that there is a real number c such that $f(x) = 100$.

(4) Show that the equation $x^5 - 3x^4 - 2x^3 - x + 1 = 0$ has at least one solution between 0 and 1.

(5) Show that the equation $x + \sin x = 1$ has at least one solution in the interval $[0, \pi/6]$.

(6) Show that the equation $x^5 + 10x + 3 = 0$ has exactly one real solution.

(7) Show that a polynomial of degree three has at most three real roots.

2. Derivatives and differentiability

(1) Verify $f(x) = x^3 + 2x + 1$ is differentiable at all points and work out the derivative.

(2) Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ and let $\beta, \gamma \in \mathbb{R}$. Let $c \in [a, b]$ and assume that $f'(c)$ and $g'(c)$ exist. Prove that $(\beta f + \gamma g)'(c) = \beta f'(c) + \gamma g'(c)$.

(3) Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ and let $c \in [a, b]$. Assume that $f'(c)$ and $g'(c)$ exist. Prove that $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$.

(4) Let $f : [a, b] \to \mathbb{R}$ be given by $f(x) = x$ and let $c \in [a, b]$. Prove that $f'(c) = 1$.

(5) Let $f : [a, b] \to \mathbb{R}$ and let $c \in [a, b]$. Prove that if $f'(c)$ exists then f is continuous at $x = c$.

(6) Prove that $\exp'(x) = \exp(x)$.

(7) Discuss the differentiability of Heavisides's step function $H(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x < 0. \end{cases}$

(8) Carefully state the chain rule and prove it.

(9) Find derivatives of all orders of $f(x) = x^k$, for $k \in \mathbb{Z}_{>0}$.

(10) Discuss the existence of, and evaluate where possible, the first and second derivatives for the function $f(x) = \begin{cases} 1 + x, & \text{if } x < 0, \\ 1 + x + x^2, & \text{if } x \geq 0. \end{cases}$.

(11) Prove that if $\alpha \in \mathbb{Q}$ and $f(x) = x^\alpha$ then $f'(x) = \alpha x^{\alpha - 1}$.

(12) Give an example of a function with a local minimum at $x = 0$.

(13) Give an example of a function with a local maximum at $x = 0$.

(14) Give an example of a function with a stationary point at $x = 0$ that is neither a local maximum or a local minimum.

(15) Prove that if f is differentiable on $[a, b]$ with $f'(a) < 0$ and $f'(b) > 0$ then there exists a point $c \in (a, b)$ at which $f'(c) = 0$. Do not assume that $f'(x)$ is continuous.

(16) Let $\epsilon \in \mathbb{R}_{>0}$. Find an interval around $x = 0$ with $|\cos x - 1| < \epsilon$.

(17) Give a simple bound for $\cos x - \cos y$.

(18) Use derivatives to prove that if $x \in \mathbb{R}$ and $x > 0$ then $x - \frac{x^3}{6} < \sin x < x$. Use this to show that $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

(19) Use derivatives to prove that if $x \in \mathbb{R}$ and $x > 0$ then $1 - \frac{x^2}{2} < \cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24}$.

(20) Let $a, b \in \mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$ be a function. Let $c \in [a, b]$. Carefully define $f'(c)$.

(21) Let $f : \mathbb{R}_{>0} \to \mathbb{R}$ be such that f is differentiable at $x = 1$ and if $x, y \in \mathbb{R}_{>0}$ then $f(xy) = f(x) + f(y)$. Show that

(a) if $c \in \mathbb{R}_{>0}$ then f is differentiable at $x = c$,
(b) if $c \in \mathbb{R}_{>0}$ then $f'(c) = f'(1)/c$,
(c) Show that f is infinitely differentiable.

(22) Let $f : \mathbb{R} \to \mathbb{R}$ be such that f is differentiable at $x = 0$ and if $x, y \in \mathbb{R}$ then $f(x + y) = f(x)f(y)$. Show that

(a) if $c \in \mathbb{R}$ then f is differentiable at $x = c$,
(b) if $c \in \mathbb{R}_{>0}$ then $f'(c) = f'(0)f(c)$,
(c) Show that f is infinitely differentiable.

(23) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \begin{cases} -x^2, & \text{if } x \le 0, \\ x, & \text{if } x > 0. \end{cases}$

Is f continuous at $x = 0$? Is f differentiable at $x = 0$?

(24) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \begin{cases} -x^2, & \text{if } x \le 0, \\ x^3, & \text{if } x > 0. \end{cases}$

Is f continuous at $x = 0$? Is f differentiable at $x = 0$?

(25) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x < 0, \\ 1 + x^2, & \text{if } x \ge 0. \end{cases}$

Is f continuous at $x = 0$? Is f differentiable at $x = 0$?

(26) Let $a, b \in \mathbb{R}$ and assume that $f : [a, b) \to \mathbb{R}$ is differentiable on (a, b) and continuous on $[a, b)$. Assume that the limit $\lim_{x \to a+} f'(x) = L$ exists. Prove that the right derivative $f_+'(a)$ exists and that $f_+'(a) = L$.

(27) Let $a, b \in \mathbb{R}$ and assume that $f : (a, b) \to \mathbb{R}$ is differentiable at c. Show that $\lim_{h \to 0+} \dfrac{f(c+h) - f(c-h)}{2h}$ exists and equals $f'(c)$. Is the converse true?

(28) Prove that $\frac{d}{dx}\sqrt{x} = \dfrac{1}{2\sqrt{x}}$.

(29) Prove that $\frac{d}{dx}\arcsin x = \dfrac{1}{\sqrt{1 - x^2}}$.

3. Rolle's Theorem

(1) State Rolle's theorem and draw a picture which illustrates the statement of the theorem.

(2) State the mean value theorem and draw a picture which illustrates the statement of the theorem.

(3) Explain why Rolle's theorem is a *special case* of the mean value theorem.

(4) Verify Rolle's theorem for the function $f(x) = (x-1)(x-2)(x-3)$ on the interval $[1, 3]$.

(5) Verify Rolle's theorem for the function $f(x) = (x-2)^2(x-3)^6$ on the interval $[2, 3]$.

(6) Verify Rolle's theorem for the function $f(x) = \sin x - 1$ on the interval $[\pi/2, 5\pi/2]$.

(7) Verify Rolle's theorem for the function $f(x) = e^{-x}\sin x$ on the interval $[0, \pi]$.

(8) Verify Rolle's theorem for the function $f(x) = x^3 - 6x^2 + 11x - 6$.

(9) Let $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but there is no number c in the interval $[-1, 1]$ such that $\left.\frac{df}{dx}\right|_{x=c} = 0$. Why does this not contradict Rolle's theorem?

(10) Let $f(x) = (x-1)^{-2}$. Show that $f(0) = f(2)$ but there is no number c in the interval $[0, 2]$ such that $\left.\frac{df}{dx}\right|_{x=c} = 0$. Why does this not contradict Rolle's theorem?

(11) Discuss the applicability of Rolle's theorem when $f(x) = (x-1)(2x-3)$ on the interval $1 \le x \le 3$.

(12) Discuss the applicability of Rolle's theorem when $f(x) = 2 + (x-1)^{2/3}$ on the interval $0 \le x \le 2$.

(13) Discuss the applicability of Rolle's theorem when $f(x) = \lfloor x \rfloor$ on the interval $-1 \le x \le 1$.

(14) At what point on the curve $y = 6 - (x-3)^2$ on the interval $[0, 6]$ is the tangent to the curve parallel to the x-axis?

4. Mean value theorem

(1) Verify the mean value theorem for the function $f(x) = x^{2/3}$ on the interval $[0, 1]$.

(2) Verify the mean value theorem for the function $f(x) = \ln x$ on the interval $[1, e]$.

(3) Verify the mean value theorem for the function $f(x) = x$ on the interval [a, b], where a and b are constants.

(4) Verify the mean value theorem for the function $f(x) = lx^2 + mx + n$ on the interval [a, b], where l, m, n, a and b are constants.

(5) Show that the mean value theorem is not applicable to the function $f(x) = |x|$ in the interval [-1, 1].

(6) Show that the mean value theorem is not applicable to the function $f(x) = 1/x$ in the interval [-1, 1].

(7) Find the points on the curve $y = x^3 - 3x$ where the tangent is parallel to the chord joining (1, -2) and (2, 2).

(8) If $f(x) = x(1 - \ln x)$, $x > 0$, show that $(a - b)\ln c = b(1 - \ln b) - a(1 - \ln a)$, for some $c \in [a, b]$ where $0 < a < b$.

(9) Let $f(x) = x^2 + 2x - 1$ and let $a = 0$ and $b = 1$. Find all values c in the interval (a, b) that satisfy the equation $f(b) - f(a) = f'(c)(b - a)$.

(10) Let $f(x) = x^3$ and let $a = 0$ and $b = 3$. Find all values c in the interval (a, b) that satisfy the equation $f(b) - f(a) = f'(c)(b - a)$.

(11) Let $f(x) = x^{2/3}$ and let $a = 0$ and $b = 1$. Find all values c in the interval (a, b) that satisfy the equation $f(b) - f(a) = f'(c)(b - a)$.

(12) Use the mean value theorem to show that if $x, y \in \mathbb{R}$ then $|\sin x - \sin y| \leq |x - y|$.

(13) Use the mean value theorem to show that if $x, y \in [2, \infty)$ then $|\log x - \log y| \leq \frac{1}{2}|x - y|$.

(14) Use the mean value theorem to show that if $x, y \in [1, \infty)$ then $|\log x - \log y| \leq |x - y|$.

(15) Use the mean value theorem to show that if $x \in \mathbb{R}_{>0}$ then $0 < (x + 1)^{1/5} - x^{1/5} < \left(5x^{4/5}\right)^{-1}$. Find $\lim_{x \to \infty} \left((x + 1)^{1/5} - x^{1/5}\right)$.

(16) Use the mean value theorem to show that if $x \in \mathbb{R}_{>1}$ then $0 < \log(x + \sqrt{x}) - \log x < x^{-1/2}$. Find $\lim_{x \to \infty} \left(\log\left(x + \sqrt{x}\right) - \log x\right)$.

(17) Use the mean value theorem to show that if a function $f : (a, b) \to \mathbb{R}$ is differentiable with $f'(x) > 0$ for all x then f is strictly increasing.

(18) Use the mean value theorem to show that if a function $f : (a, b) \to \mathbb{R}$ is twice differentiable with $f''(x) > 0$ then f is strictly convex. (A function f is strictly convex if $f(tx + (1 - t)y) < tf(x) + (1 - t)f(y)$ for all $x, y \in (a, b)$ and $t, y \in (0, 1)$.

5. Taylor approximation

(1) Compare $f(x) = \sqrt{2}\sin x$ with its fifth order Taylor polynomial about $x = \pi/4$.

(2) Discuss the Taylor polynomial approximations about $x = 0$ to $f(x) = (1+x)^{-1}$.

(3) Show how we can compute $\log(1.1)$ correct to three decimal places by a polynomial approximation.

(4) Prove that if $x \in \mathbb{R}$ and $0 \le x \le 1$ then $\log(1+x) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{(-1)^{k-1} x^k}{k}$.

(5) Use Taylor approximation to prove that if $\alpha \in \mathbb{R}_{>0}$ then $\lim_{x\to\infty} \frac{x^\alpha}{e^x} = 0$.

(6) Use Taylor approximation to prove that if $\alpha \in \mathbb{R}_{>0}$ then $\lim_{x\to\infty} \frac{\log x}{x^\alpha} = 0$.

(7) Use Taylor approximation to prove that if $\alpha \in \mathbb{R}_{>0}$ then $\lim_{x\to 0^+} x^\alpha \log x = 0$.

(8) Let $a = 0$ and $n = 4$. If possible, construct the Taylor polynomial about $x = a$ of order n for $f(x) = \sin x$. Explain clearly what has gone wrong if the Taylor polynomial cannot be constructed.

(9) Let $a = \pi/4$ and $n = 4$. If possible, construct the Taylor polynomial about $x = a$ of order n for $f(x) = \sin x$. Explain clearly what has gone wrong if the Taylor polynomial cannot be constructed.

(10) Let $a = 0$ and $n = 3$. If possible, construct the Taylor polynomial about $x = a$ of order n for $f(x) = \frac{1}{1+x}$. Explain clearly what has gone wrong if the Taylor polynomial cannot be constructed.

(11) Let $a = 1$ and $n = 3$. If possible, construct the Taylor polynomial about $x = a$ of order n for $f(x) = \frac{1}{1+x}$. Explain clearly what has gone wrong if the Taylor polynomial cannot be constructed.

(12) Let $a = 0$ and $n = 2$. If possible, construct the Taylor polynomial about $x = a$ of order n for $f(x) = \frac{1}{1+\sqrt{x}}$. Explain clearly what has gone wrong if the Taylor polynomial cannot be constructed.

(13) Let $a = 1$ and $n = 2$. If possible, construct the Taylor polynomial about $x = a$ of order n for $f(x) = \frac{1}{1+\sqrt{x}}$. Explain clearly what has gone wrong if the Taylor polynomial cannot be constructed.

(14) Let $a = -1$ and $n = 3$. If possible, construct the Taylor polynomial about $x = a$ of order n for $f(x) = |x+1|^3$. Explain clearly what has gone wrong if the Taylor polynomial cannot be constructed.

(15) Let $a = 1$ and $n = 3$. If possible, construct the Taylor polynomial about $x = a$ of order n for $f(x) = |x+1|^3$. Explain clearly what has gone wrong if the Taylor polynomial cannot be constructed.

(16) Let $a = 0$ and $n = 2$. If possible, construct the Taylor polynomial about $x = a$ of order n for $f(x) = \begin{cases} \sqrt{1-x^2}, & \text{if } 0 \le x < 1, \\ \cos x, & \text{if } x < 0. \end{cases}$
Explain clearly what has gone wrong if the Taylor polynomial cannot be constructed.

(17) Use derivatives to derive the Taylor polynomial for $f(x) = \exp x$ about $x = 0$.

(18) Use derivatives to derive the Taylor polynomial for $f(x) = \sin x$ about $x = 0$.

(19) Use derivatives to derive the Taylor polynomial for $f(x) = \cos x$ about $x = 0$.

(20) Use derivatives to derive the Taylor polynomial for $f(x) = \log(1+x)$ about $x = 0$.

(21) Use derivatives to derive the Taylor polynomial for $f(x) = \log(1-x)$ about $x = 0$.

(22) Using the remainder estimate from Taylor's theorem, determine a bound on the error in approximating cosh 1 by the degree 8 Taylor polynomial about $x = 0$ for $\cosh x$. You may use the facts: $\sinh 1 < \cosh 1 < 3$ and $9! \approx 3.6 \cdot 10^5$.

(23) Using the remainder estimate from Taylor's theorem, determine a bound on the error in approximating sinh 1 by the degree 9 Taylor polynomial about $x = 0$ for $\sinh x$. You may use the facts: $\sinh 1 < \cosh 1 < 3$ and $10! \approx 3.6 \cdot 10^6$.

(24) Write down the degree 5 Taylor polynomial for $f(x) = \sin x$. Use Taylor's theorem to write down an expression for the error $R_5(x)$, where you may assume that $0 < x < \pi/2$. In what interval does the unknown constant c lie? Hence show that

$$\frac{x^6}{6!} < R_5(x) < 0.$$

Use this inequality and

$$\sin x = P_5(x) + R_5(x)$$

to find upper and lower bounds for $\sin x$ in terms of $P_5(x)$.

(25) Let $a = 1$ and $n = 4$. If possible, construct the Taylor polynomial about $x = a$ of order n for $f(x) = \sqrt{x}$. Explain clearly what has gone wrong if the Taylor polynomial cannot be constructed.

(26) Let $a = 1$ and $n = 4$. If possible, construct the Taylor polynomial about $x = a$ of order n for $f(x) = \dfrac{1}{x}$. Explain clearly what has gone wrong if the Taylor polynomial cannot be constructed.

(27) Let $a = 1$ and $n = 4$. If possible, construct the Taylor polynomial about $x = a$ of order n

for $f(x) = \tan x$. Explain clearly what has gone wrong if the Taylor polynomial cannot be constructed.

(28) Let $a = 0$ and $n = 3$ and let $x \in \mathbb{R}$ with $-1 \le x \le 1$. Let $f(x) = \cos x$. Construct the Taylor polynomial for $f(x)$ of order n about $x = a$ and find a close bound for $|R_n(x)|$, where $R_n(x) = f(x) - P_n(x)$.

(29) Let $a = 0$ and $n = 2$ and let $x \in \mathbb{R}$ with $-0.5 \le x \le 0.5$. Let $f(x) = e^x$. Construct the Taylor polynomial for $f(x)$ of order n about $x = a$ and find a close bound for $|R_n(x)|$, where $R_n(x) = f(x) - P_n(x)$.

(30) Let $a = \pi/4$ and $n = 5$ and let $x \in \mathbb{R}$ with $0 \le x \le \pi/2$. Let $f(x) = \sin x$. Construct the Taylor polynomial for $f(x)$ of order n about $x = a$ and find a close bound for $|R_n(x)|$, where $R_n(x) = f(x) - P_n(x)$.

(31) Let $a = 0$ and $n = 4$ and let $x \in \mathbb{R}$ with $0 \le x \le 1$. Let $f(x) = \sinh x$. Construct the Taylor polynomial for $f(x)$ of order n about $x = a$ and find a close bound for $|R_n(x)|$, where $R_n(x) = f(x) - P_n(x)$.

(32) Use Taylor polynomials to approximate $\sqrt{e}$ to four decimal places.

(33) Use Taylor polynomials to approximate e^{-1} to four decimal places.

(34) Use Taylor polynomials to approximate log 1.5 to four decimal places.

(35) Use Taylor polynomials to approximate sinh 0.5 to four decimal places.

(36) Let $a = \pi/4$ and $n = 5$ and let $x \in \mathbb{R}$ with $0 \le x \le \pi/2$. Let $f(x) = \sin x$. Construct the Taylor polynomial for $f(x)$ of order $n$about $x = a$ and find a close bound for $|R_n(x)|$, where $R_n(x) = f(x) - P_n(x)$. Use this information to estimate $\sin 35^\circ$ to five decimal places.

(37) For what values of x can we replace $\sqrt{1 + x}$ by $1 + \frac{1}{2}x$ with an error of less than 0.01?

(38) Write down a polynomials approximation for $f(x) = \sin x$ at $x = 0$. How many terms do you need for the approximation to be correct to three decimal places if $|x| < 0.5$?

(39) An electric dipole on the x-axis consists of a charge Q at $x = 1$ and a charge $-Q$ at $x = -1$. The electric field E at the point $x = R$ on the x-axis is given (for $R > 1$) by

$$E = \frac{kQ}{(R-1)^2} - \frac{kQ}{(R+1)^2},$$

where k is a positive constant whose value depends on the units. Expand E as a series in $\frac{1}{R}$, giving the first two nonzero terms.

(40) Write a quadratic approximation for $f(x) = x^{1/3}$ near 8 and approximate $9^{1/3}$. Estimate the error and find the smallest interval that you can be sure contains the value.

(41) Write a quadratic approximation for $f(x) = x^{-1}$ near 1 and approximate 1/1.02. Estimate the error and find the smallest interval that you can be sure contains the value.

(42) Write a quadratic approximation for $f(x) = e^x$ near 0 and approximate $e^{-0.5}$. Estimate the error and find the smallest interval that you can be sure contains the value.

(43)

(a) From Taylor's theorem write down an expansion for the remainder when the Taylor polynomial of degree N for e^x (about $x = 0$) is subtracted from e^x. In what interval does the unknown constant c lie, if $x > 0$?

(b) Show that if $x > 0$ then the remainder has the bounds $\frac{x^{n+1}}{(n+1)!} < R_N < e^x \frac{x^{n+1}}{(n+1)!}$ and use the sandwich rule to show that $R_N \to 0$ as $N \to \infty$. This proves that the Taylor series for e^x does converge to e^x, for any $x > 0$.

6. References

[Ca] S. Carnie, *620-143 Applied Mathematics, Course materials*, 2006 and 2007.

[Ho] C. Hodgson, *620-194 Mathematics B and 620-211 Mathematics 2 Notes*, Semester 1, 2005.

[Hu] B.D. Hughes, *620-158 Accelerated Mathematics 2 Lectures*, 2009.