Problem Set - Series
620-295 Semester I 2010

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Last updates: 10 March 2010

1. Series

For each of the following series

(a) Write out the first five terms of the series,
(b) Write out the first five partial sums,
(c) Determine if the series converges,
(d) Apply the ratio test if applicable,
(e) Apply the root test, if applicable,
(f) Apply the integral test, if possible,
(g) determine if the series converges absolutely or conditionally,
(h) Apply the alternating series test, if appropriate.

Powers of n

(1) $$\sum_{n=1}^{\infty} \frac{1}{n^5}$$

(2) $$\sum_{n=1}^{\infty} \frac{1}{n^2+4}$$

(3) $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

(4) $\sum_{n=2}^{\infty} \frac{1}{(n-1)^2}$

(5) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

(6) $\sum_{n=2}^{\infty} \frac{n}{n^3-1}$

(7) $\sum_{n=1}^{\infty} \frac{1}{n+1}$

(8) $\sum_{n=2}^{\infty} \frac{1}{n-1}$

(9) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

(10) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$

(11) $\sum_{n=1}^{\infty} \sqrt{\frac{n}{n+1}}$

(12) $\sum_{n=1}^{\infty} \frac{1}{n^7}$

(13) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}}$

(14) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$

(15) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{1+n^2}$

(16) $\displaystyle\sum_{n=1}^{\infty} \frac{2+\cos n}{n^2}$

(17) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$

(18) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 nx}{n^2}$

(19) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{\sqrt[3]{n^2+2}}$

(20) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(4n-3)(4n+1)}$

(21) $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$

(22) $\displaystyle\sum_{n=1}^{\infty} \frac{3n}{5n+1}$

(23) $\displaystyle\sum_{n=1}^{\infty} a_n$, where $a_n = \frac{2n}{3n+1}$.

(24) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

(25) $\sum_{n=1}^{\infty} \frac{n}{n+1}$

(26) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$

(27) $\sum_{n=1}^{\infty} \frac{3n^2-2}{n^4+8n}$

(28) $\sum_{n=1}^{\infty} \frac{2n}{n^2+1}$

(29) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+2}$

(30) $\sum_{n=1}^{\infty} \frac{3}{\sqrt[3]{n^2+2}}$

(31) $\sum_{n=1}^{\infty} \frac{n^3+4n}{n^4+200}$

(32) $\sum_{n=1}^{\infty} \frac{4n^2-2}{3-5n^2}$

(33) $\sum_{n=1}^{\infty} \frac{\sqrt{n}+3}{2n^2\sqrt{n}+7}$

n in the exponent

(1) $\sum_{n=1}^{\infty} \frac{3+e^{-n}}{2n^{2/3}-1}$

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^n + 1}$

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{5^n}$

(4) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n + 1}{4^n + 1}$

(5) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{2^n}$

(6) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{4^n}$

(7) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n + 7n}{2^n(n^2 + 1)}$

(8) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n + 1}$

(9) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

(10) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$

(11) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n-1}{n}\right)^n$

(12) $\displaystyle\sum_{n=1}^{\infty} \frac{10}{3^n}$

(13) $\sum_{n=1}^{\infty} \frac{1}{3^n}$

(14) $\sum_{n=1}^{\infty} 2^{-n} 3^{n-1}$

(15) $\sum_{n=1}^{\infty} \frac{1}{n4^n}$

(16) $\sum_{n=1}^{\infty} \frac{1}{1+3^n}$

(17) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

(18) $\sum_{n=1}^{\infty} \frac{n^2}{e^n}$

(19) $\sum_{n=1}^{\infty} \frac{n^3}{2^n}$

(20) $\sum_{n=1}^{\infty} \frac{2^n}{n+1}$

Alternating series

(1) $\sum_{n=1}^{\infty} (-1)^n \frac{2n}{4n^2-3}$

(2) $\sum_{n=1}^{\infty} (-1)^n \frac{3}{5^n}$

(3) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2n}{4n-3}$

(4) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$

(5) $\frac{2}{1} - \frac{2}{2} + \frac{2}{3} - \frac{2}{4} + \frac{2}{5} - \cdots$

(6) $-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \cdots$

(7) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\log(n+1)}$

(8) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$

(9) $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^n}{n!}$

(10) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{(-2)^k}{7^k} + \frac{1}{4^{k-1}} \right)$

(11) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

(12) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

(13) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{2n+1}}$

(14) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{n}$

(15) $\displaystyle\sum_{n=1}^{\infty} (-1)^n$

(16) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1+3^n}{1+4^n}$

(17) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$

(18) $1 - 3 + 9 - 27 + \cdots$

Other series

(1) $\log \frac{1}{2} + \log \frac{2}{3} + \log \frac{3}{4} + \cdots$

(2) $\displaystyle\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots$

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

(4) $\displaystyle\sum_{n=1}^{\infty} \log\left(\frac{(n+1)^2}{n(n+2)}\right)$

(5) $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt[n]{n+3} - \sqrt[n-1]{n+2}\right)$

(6) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{2^n} - \frac{5}{3^n}\right)$

(7) $$\sum_{n=1}^{\infty}\left(\frac{1}{n^{1.5}}-\frac{1}{(n+1)^{1.5}}\right)$$

(8) $$\sum_{n=1}^{\infty}\log\left(\frac{n}{n+1}\right)$$

(9) $$\sum_{n=1}^{\infty}\left(\exp\left(\frac{n+1}{n}\right)-\exp\left(\frac{n+2}{n+1}\right)\right)$$

2. Radius of convergence

For each of the following series find the set of $x \in \mathbb{C}$ such that the series converges. Find the radius of convergence and the interval of convergence.

(1) $$\sum_{n=0}^{\infty}\frac{x^n}{n+1}$$

(2) $$\sum_{n=0}^{\infty}(-1)^n\frac{(x+1)^n}{(n+1)^2}$$

(3) $$\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

(4) $$\sum_{n=1}^{\infty}\frac{(2x-1)^n}{\sqrt[3]{n}}$$

(5) $$\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$$

(6) $$\sum_{n=0}^{\infty}\frac{2}{2n+1}x^{2n+1}$$

(7) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

(8) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{2^{n+1}x^{n+1}}{n+1}$

(9) $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^k x^k}{k}$

(10) $\displaystyle\sum_{n=1}^{\infty} nx^{n-1}$

(11) $\displaystyle\sum_{n=2}^{\infty} n(n-1)x^{n-2}$

(12) $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^k x^k}{k}$

(13) $\displaystyle\sum_{n=1}^{\infty} nx^n$

(14) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$

(15) $\displaystyle\sum_{n=2}^{\infty} n(n-1)x^n$

(16) The power series expansion of $\sin(2x)$

(17) The power series expansion of $\cos x$

(18) The power series expansion of $\dfrac{1}{1+x}$

(19) The power series expansion of $\sinh x$

(20) The power series expansion of $\log(2x+1)$

(21) The power series expansion of $(1+x)^{-2}$

(22) The power series expansion of $\sin(\theta^2)$

(23) The power series expansion of $x \sin 3x$

(24) The power series expansion of $\cos^2 x$

(25) The power series expansion of $\dfrac{t}{1+t}$

(26) The power series expansion of $\dfrac{z}{e^{2z}}$

(27) The power series expansion of $\sin(x^2)$

(28) The power series expansion of $\int_0^t \sin x^2 \quad dx$

(29) The power series expansion of $\displaystyle\int \frac{\sinh x}{x} \quad dx$

(30) The power series expansion of $\displaystyle\int \frac{\cosh x - 1}{x^2} \quad dx$

(31) The Taylor series for $\sin x$ at the point $a = \frac{1}{4}\pi$

(32) The Taylor series for $\cos x$ at the point $a = \frac{1}{3}\pi$

(33) The Taylor series for $\frac{1}{x}$ at the point $a = 2$.

(34) The Taylor series for e^x at the point $a = -3$.

(35) The series representation for e^{2x} in powers of $x+1$

(36) The series representation for $\log x$ in powers of $x-1$.

3. References

[Ca] S. Carnie, *620-143 Applied Mathematics, Course materials*, 2006 and 2007.

[Ho] C. Hodgson, *620-194 Mathematics B and 620-211 Mathematics 2 Notes*, Semester 1, 2005.