Problem Set: Sequences
620-295 Semester I 2010

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1. Sequence analysis

For each of the following sequences:

(a) explicitly write out the first 7 terms,
(b) graph the sequence,
(c) determine if it is bounded,
(d) determine if it is increasing or decreasing,
(e) determine if it is Cauchy,
(f) determine the sup and inf,
(g) determine the lim sup and lim inf,
(h) determine if it is convergent or divergent,
(i) determine the limit if it is convergent,
(m) determine if it is contractive.

(1) Analyse the sequence $a_n = n$.

(2) Analyse the sequence $a_n = (-1)^n n$.

(3) Analyse the sequence $a_n = n^2$.

(4) Analyse the sequence $a_n = 12n - n^3$.

(5) Analyse the sequence $a_n = n!$.

(6) Analyse the sequence $a_n = \frac{1}{n}$.

(7) Analyse the sequence $a_n = 3 - \frac{1}{n}$.

(8) Analyse the sequence $a_n = \frac{1}{n^p}$.

(9) Analyse the sequence $a_n = \frac{1}{n!}$.

(10) Analyse the sequence $a_n = \frac{n}{n(n+1)}$.

(11) Analyse the sequence $a_n = \frac{1}{n} - \frac{1}{n+1}$.

(12) Analyse the sequence $a_n = \frac{(-1)}{n+1}$.

(13) Analyse the sequence $a_n = \frac{(-1)^{n+1}}{n}$.

(14) Analyse the sequence $a_n = (-1)^n\left(1 + \frac{1}{n}\right)$.

(15) Analyse the sequence $a_n = \frac{n}{2n+1}$.

(16) Analyse the sequence $a_n = \frac{2n}{n+1}$.

(17) Analyse the sequence $a_n = \frac{n}{n^2+1}$.

(18) Analyse the sequence $a_n = \frac{3n+1}{2n+5}$.

(19) Analyse the sequence $a_n = \dfrac{n^2 - 1}{2n^2 + 3}$.

(20) Analyse the sequence $a_n = \dfrac{i^n}{n^2}$.

(21) Analyse the sequence $a_n = \dfrac{n + 2i}{n}$.

(22) Analyse the sequence $a_n = \dfrac{4n + 3}{4n^2 + 3n + 1}$.

(23) Analyse the sequence $a_k = \dfrac{1}{\left(3k^4 - 7k^2 + 5\right)^{\frac{1}{3}}}$.

(24) Analyse the sequence $a_n = \dfrac{(n!)^2}{(2n)!}$.

(25) Analyse the sequence $a_n = \dfrac{(n!)^2 5^n}{(2n)!}$.

(26) Analyse the sequence $a_n = (-1)^n$.

(27) Analyse the sequence $a_n = n^{1/n}$.

(28) Analyse the sequence $a_n = \left(1 + \dfrac{1}{n}\right)^n$.

(29) Analyse the sequence $a_n = e^{in\pi/7}$.

(30) Analyse the sequence $a_n = \sqrt{n}$.

(31) Analyse the sequence $a_n = \dfrac{1}{\sqrt{n}}$.

(32) Analyse the sequence $a_n = \sqrt{n + 1} - \sqrt{n}$.

(33) Analyse the sequence $a_n = \sqrt{n}\big(\sqrt{n+1} - \sqrt{n}\big)$.

(34) Let $x \in \mathbb{R}$ with $|x| < 1$. Analyse the sequence $a_n = x^n$.

(35) Let $x \in \mathbb{R}$ with $x > 0$. Analyse the sequence $a_n = x^{1/n}$.

(36) Let $x \in \mathbb{R}$. Analyse the sequence $a_n = \left(1 + \frac{x}{n}\right)^n$.

(37) Let $x \in \mathbb{R}$. Analyse the sequence $a_n = \dfrac{1 - x^{n+1}}{1 - x}$.

(38) Let $x \in \mathbb{R}$. Analyse the sequence $a_n = 1 + x + \cdots + x^n$.

(39) Analyse the sequence given by $a_1 = 3$ and $a_n = \dfrac{1}{2}\left(a_{n-1} + \dfrac{5}{a_{n-1}}\right)$.

(40) Let $a \in \mathbb{R}$ with $a > 0$. Fix a positive real number x_1. Analyse the sequence given by $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{a}{x_n}\right)$.

(41) Let $\alpha,\ \beta \in \mathbb{R}_{>0}$. Analyse the sequence given by $a_1 = \alpha$ and $a_{n+1} = \sqrt{\beta + a_n}$.

(42) Let $\alpha,\ \beta \in \mathbb{R}_{>0}$. Analyse the sequence given by $a_1 = \alpha$ and $a_{n+1} = \beta + \sqrt{a_n}$.

(43) Analyse the sequence given by $x_1 = 1$ and $x_{n+1} = \dfrac{1}{2 + x_n}$.

(44) Fix a real number x_1 between 0 and 1. Analyse the sequence given by $x_{n+1} = \dfrac{1}{7}\left(x_n^3 + 2\right)$. Estimate the solution to $x^3 - 7x + 2 = 0$ to three decimal places and verify that the limit is a solution to the equation $x^3 - 7x + 2 = 0$.

(45) Analyse the sequence given by $a_1 = 0,\ \ a_{2k} = \frac{1}{2}a_{2k+1},$ and $a_{2k+1} = \frac{1}{2} + a_{2k}$.

(46) Analyse the sequence $a_n = \dfrac{n}{2n+3}$.

(47) Analyse the sequence $a_n = \frac{n}{n+1} - \frac{n+1}{n}$.

(48) Analyse the sequence $a_n = \frac{1-n}{n^3}$.

(49) Analyse the sequence $a_n = \frac{3n-1}{2n+5}$.

(50) Analyse the sequence $a_n = \frac{n+1}{\sqrt{n}}$.

(51) Analyse the sequence $a_n = \frac{\sqrt{n}}{n+1}$.

(52) Analyse the sequence $a_n = 1 + (-1)^{n+1}$.

(53) Analyse the sequence $a_n = n^{(-1)^n}$.

(54) Analyse the sequence $a_n = a_n$ when $a_n = \sqrt{3}$.

(55) Analyse the sequence $a_n = \frac{n}{2^n}$.

(56) Analyse the sequence $a_n = \cos \frac{n\pi}{2}$.

(57) Analyse the sequence $a_n = (1 + (-1)^n) \frac{1}{n}$.

(58) Analyse the sequence $a_n = \frac{4n^2 - 2n + \cos n}{3n^2 + 7n + 6}$.

(59) Analyse the sequence $a_n = \frac{3n-5}{\sqrt{2n^2+1} + \sqrt{3n^2+4}}$.

(60) Analyse the sequence $a_n = \frac{2n^2 + 6n + 2}{3n^3 - n^2 - n}$.

(61) Analyse the sequence $a_n = \sqrt{n^2 + 3n} - \sqrt{n^2 + 4}$.

(62) Analyse the sequence $a_n = \frac{1}{3n+5}$.

(63) Analyse the sequence $a_n = 3 + \frac{(-1)^n}{n}$.

(64) Analyse the sequence $a_n = \frac{n-2}{n+2}$.

(65) Analyse the sequence $a_n = \left(\frac{n}{25}\right)^{1/3}$.

(66) Analyse the sequence $a_n = \dfrac{2^{2n}}{n!}$.

(67) Analyse the sequence given by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$. In particular, show that a_n is increasing and bounded above by 3.

(68) Analyse the sequence given by $a_1 = 2$ and $a_{n+1} = 3 - \dfrac{1}{n}$. In particular, show that a_n is increasing and $a_n < 3$ for all n.

(69) Suppose the nth pass through a manufacturing process is modelled by the linear equations $x_n = A^n x_0$, where x_0 is the initial state of the system and $A = \frac{1}{5}\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$. Show that

$$A^n = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \left(\frac{1}{5}\right)^n \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Then with the initial state $x_0 = \begin{pmatrix} p \\ 1-p \end{pmatrix}$, calculate $\lim_{n\to\infty} x_n$.

(70) The Fibbonaci sequence 0,1,1,2,3,5,8,13, ... is described by the difference equation $F_{k+2} = F_{k+1} + F_k$ and the initial conditions $F_0 = 0$, $F_1 = 1$. Writing $u_k = \begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix}$, show that $u_{k+1} = Au_k$, where

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Solve for u_k in terms of $u_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and show that

$$F_k = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right),$$

and therefore find the limit as $k \to \infty$ of the ratio F_{k+1}/F_k.

2. Definitions and proofs

(1) What is a sequence?

(2) What is a convergent sequence?

(3) What is a divergent sequence?

(4) What is the limit of a sequence?

(5) What is the sup of a sequence?

(6) What is the inf of a sequence?

(7) What is the lim sup of a sequence?

(8) What is the lim inf of a sequence?

(9) What is a bounded sequence?

(10) What is an increasing sequence?

(11) What is a decreasing sequence?

(12) What is a monotone sequence?

(13) What is a Cauchy sequence?

(14) What is a contractive sequence?

(15) Prove that if (a_n) is a sequence in $\mathbb{C}$ and (a_n) converges then $\lim_{n\to\infty} a_n$ is unique.

(16) Prove that if (a_n) is a sequence in $\mathbb{C}$ and (a_n) converges then (a_n) is bounded.

(17) Prove that if (a_n) and (b_n) are sequences in $\mathbb{C}$ and $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ then $\lim_{n\to\infty} a_n + b_n = a + b$.

(18) Prove that if (a_n) and (b_n) are sequences in $\mathbb{C}$ and $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ then $\lim_{n\to\infty} a_n b_n = ab$.

(19) Prove that if (a_n) and (b_n) are sequences in $\mathbb{C}$ and $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ and $b_n \neq 0$ for all $n \in \mathbb{Z}_{>0}$ then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$.

(20) Prove that if (a_n), (b_n) and (c_n) are sequences in $\mathbb{R}$ and $\lim_{n\to\infty} a_n = \ell$ and $\lim_{n\to\infty} c_n = \ell$ and $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{Z}_{>0}$ then $\lim_{n\to\infty} b_n = \ell$.

(21) Prove that if (a_n), is a sequence in $\mathbb{R}$ and (a_n) is increasing and bounded above then (a_n) converges.

(22) Prove that if (a_n), is a sequence in $\mathbb{R}$ and (a_n) is not bounded then (a_n) diverges.

(23) Find a sequence a_n in $\mathbb{R}$ such that if $r \in [0, 1]$ there is a subsequence of a_n which converges to r.

3. Picard and Newton iteration

(1) Let $f : (0, \frac{1}{2}\pi) \to \mathbb{R}$ is given by $f(x) = \frac{1}{2}\tan x$. Estimate numerically the solution to $x = f(x)$ with $x \in (0, \frac{1}{2}\pi)$ using Picard iteration.

(2) Let $f : (0, \frac{1}{2}\pi) \to \mathbb{R}$ is given by $f(x) = \frac{1}{2}\tan x$. Estimate numerically the solution to $x = f(x)$ with $x \in (0, \frac{1}{2}\pi)$ using Newton iteration (let $F(x) = x - f(x)$).

(3) Show that the equation $g(x) = x^3 + x - 1 = 0$ has a solution between 0 and 1. Transform the equation to the form $x = f(x)$ for a suitable function $f : [0, 1] \to [0, 1]$. Use Picard iteration to find the solution to 3 decimal places. (Try $f(x) = 1/(x^2 + 1)$).

(4) Show that the equation $g(x) = x^4 - 4x^2 - x + 4 = 0$ has a solution between $\sqrt{3}$ and 2. Transform the equation to the form $x = f(x)$ for a suitable function $f : [\sqrt{3}, 2] \to [\sqrt{3}, 2]$. Use Picard iteration to find the solution to 3 decimal places. (Try $f(x) = \sqrt{2 + \sqrt{x}}$).

(5) Applying Newton's method to solve the equation $f(x) = x^2 - 2 = 0$ gives a sequence x_n defined recursively by $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, for each $n \geq 1$. We choose $x_1 = 2$ as our initial approximate solution.

(a) Verify that $x_{n+1} = \frac{x_n}{1} + \frac{1}{x_n}$. Use this to calculate x_2 and x_3.
(b) Show that if the limit $\lim_{n\to\infty} x_n = L$ exists, then it must satisfy $L^2 - 2 = 0$.
(c) Show, by induction on , that $\sqrt{2} < x_n \leq 2$, for all n.
(d) Show that $x_{n+1} < x_n$, for all n.
(e) Deduce that the sequence x_n has a limit, and that $\lim_{n\to\infty} x_n = \sqrt{2}$.

4. References

[Ca] S. Carnie, *620-143 Applied Mathematics, Course materials*, 2006 and 2007.

[Ho] C. Hodgson, *620-194 Mathematics B and 620-211 Mathematics 2 Notes*, Semester 1, 2005.

[Wi] P. Wightwick, *UMEP notes*, 2010.