Problem Sheet -- Expressions
620-295 Semester I 2010

Arun Ram
Department of Mathematics and Statistics
University of Melbourne
Parkville, VIC 3010 Australia
aram@unimelb.edu.au
and

Department of Mathematics
University of Wisconsin, Madison
Madison, WI 53706 USA
ram@math.wisc.edu

Last updates: 28 February 2010

1. Expressions

"What is" is a synonym of "Define".

(1) What is x^2?

(2) What is e^x?

(3) What is $\sin x$?

(4) What is $\cos x$?

(5) What is $\tan x$?

(6) What is $\cot x$?

(7) What is $\sec x$?

(8) What is $\csc x$?

(9) What is $\sinh x$?

(10) What is $\cosh x$?

(11) What is $\tanh x$?

(12) What is coth x?

(13) What is sech x?

(14) What is csch x?

2. Inverse Expressions

(1) What is $\sqrt{x}$?

(2) What is $x^{1/2}$?

(3) What is ln x?

(4) What is log x?

(5) What is $\sin^{-1} x$?

(6) What is arcsin x?

(7) What is $\cos^{-1} x$?

(8) What is arccos x?

(9) What is $\tan^{-1} x$?

(10) What is arctan x?

(11) What is $\cot^{-1} x$?

(12) What is arccot x?

(13) What is $\sec^{-1} x$?

(14) What is arcsec x?

(15) What is $\csc^{-1} x$?

(16) What is arccsc x?

(17) What is $\sinh^{-1} x$?

(18) What is arcsinh x?

(19) What is $\cosh^{-1} x$?

(20) What is arccosh x?

(21) What is $\tanh^{-1} x$?

(22) What is arctanh x?

(23) What is $\coth^{-1} x$?

(24) What is arccoth x?

(25) What is $\operatorname{sech}^{-1} x$?

(26) What is arcsech x?

(27) What is $\operatorname{csch}^{-1} x$?

(28) What is arccsch x?

3. Expression identities

"Explain why" is a synonym for "Prove that". "Verify the identity" is another synonym for "Prove that".

(1) Explain why $\dfrac{1}{1-\mathrm{x}} = 1 + x + x^2 + x^3 + \cdots$.

(2) Explain why $\dfrac{x^n - 1}{x - 1} = 1 + x + x^2 + \cdots + x^{n-1}$.

(3) Find all possibilities for $c_0, c_1, c_2, \ldots$ so that $f(x) = c_0 + c_1 x + c_2 x^2 + \cdots$ satisfies $f(x + y) = f(x)f(y)$.

(4) Explain why $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dfrac{x^5}{5!} + \dfrac{x^6}{6!} + \cdots$.

(5) Explain why $\ln x$ is the inverse function to e^x.

(6) Verify the identity $e^{x+y} = e^x e^y$.

(7) Verify the identity $e^{-x} = \dfrac{1}{e^x}$.

(8) Verify the identity $(e^x)^n = e^{nx}$.

(9) Verify the identity $e^0 = 1$.

(10) Verify the identity $\ln(xy) = \ln x + \ln y$.

(11) Verify the identity $-\ln x = \ln(1/\mathrm{x})$.

(12) Verify the identity $\ln x^n = n \ln x$.

(13) Verify the identity $\ln 1 = 0$.

(14) Explain why $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$.

(15) Explain why $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$.

(16) Verify the identity $e^{ix} = \cos x + i \sin x$.

(17) Verify the identity $\cos^2 x + \sin^2 x = 1$.

(18) Verify the identity $\sin(-x) = -\sin x$.

(19) Verify the identity $\cos(-x) = \cos x$.

(20) Verify the identity $\sin(x + y) = \sin x \cos y + \cos x \sin y$.

(21) Verify the identity $\cos(x + y) = \cos x \cos y - \sin x \sin y$.

(22) Verify the identity $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$.

(23) Verify the identity $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$.

(24) Explain why $\cosh x = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \cdots$.

(25) Explain why $\sinh x = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dfrac{x^7}{7!} + \cdots$.

(26) Verify the identity $e^x = \cosh x + \sinh x$.

(27) Verify the identity $\cosh^2 x - \sinh^2 x = 1$.

(28) Verify the identity $\sinh(-x) = -\sinh x$.

(29) Verify the identity $\cosh(-x) = \cosh x$.

(30) Verify the identity $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$.

(31) Verify the identity $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$.

(32) Verify the identity $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

(33) Verify the identity $\sinh x = \dfrac{e^x + e^{-x}}{2}$.

(34) Verify the identity $\operatorname{arcsinh} x = \log\left(x + \sqrt{x^2 + 1}\right)$.

(35) Verify the identity $\operatorname{arccosh} x = \log\left(x + \sqrt{x^2 - 1}\right)$.

(36) Verify the identity $\operatorname{arctanh} t = \frac{1}{2}\log\left(\frac{1+t}{1-t}\right)$.

4. Trig identities

(1) Verify the identity $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$.

(2) Verify the identity $\sin(x/2) = \pm\sqrt{\frac{1-\cos x}{2}}$.

(3) Verify the identity $\cos 3x = \cos^3 x - 3\cos x \sin^2 x$.

(4) Verify the identity $\sin 3x = 3\cos^2 x \sin x - \sin^3 x$.

(5) Verify the identity $\sin^2 A \cot^2 A = (1 - \sin A)(1 + \sin A)$.

(6) Verify the identity $\tan B = \frac{\cos B}{\sin B \cot^2 B}$.

(7) Verify the identity $\frac{\tan V \cos V}{\sin V} = 1$.

(8) Verify the identity $\sin E \cot E + \cos E \tan E = \sin E + \cos E$.

(9) Verify the identity $\frac{1}{\sec^2 x} + \frac{1}{\csc^2 x} - 1 = 0$.

(10) Verify the identity $\frac{\sec A - 1}{\sec A + 1} + \frac{\cos A - 1}{\cos A + 1} = 0$.

(11) Verify the identity $\sin V\left(1 + \cot^2 V\right) = \csc V$.

(12) Verify the identity $\frac{\sin(\pi/2 - w)}{\cos(\pi/2 - w)} = \cot w$.

(13) Verify the identity $\sec(\pi/2 - z) = \frac{1}{\sin z}$.

(14) Verify the identity $1 + \tan^2(\pi/2 - x) = \frac{1}{\cos^2(\pi/2 - x)}$.

(15) Verify the identity $\frac{\sin A}{\csc A} + \frac{\cos A}{\sec A} = 1$.

(16) Verify the identity $\frac{\sec B}{\cos B} - \frac{\tan B}{\cot B} = 0$.

(17) Verify the identity $\frac{1}{\csc^2 w} + \sec^2 w + \frac{1}{\sec^2 w} = 2 + \frac{\sec^2 w}{\csc^2 w}$.

(18) Verify the identity $\sec^4 V - \sec^2 V = \dfrac{1}{\cot^4 V} + \dfrac{1}{\cot^2 V}$.

(19) Verify the identity $\sin^4 x + \cos^2 x = \cos^4 x + \sin^2 x$.

(20) Verify the identity $\tan 3\alpha = \dfrac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$.

(21) Verify the identity $\cot(\alpha/2) = \dfrac{\sin\alpha}{1 - \cos\alpha}$.

(22) Verify the identity $\cos(\pi/6 - x) + \cos(\pi/6 + x) = \sqrt{3}\cos x$.

(23) Verify the identity $\sin(\alpha + \beta)\sin(\alpha - \beta) = \sin^2\alpha - \sin^2\beta$.

(24) Verify the identity $\sin(\pi/3 - x) + \sin(\pi/3 + x) = \sqrt{3}\cos x$.

(25) Verify the identity $\cos(\pi/4 - x) - \cos(\pi/4 + x) = \sqrt{2}\sin x$.

(26) Verify the identity $2\sin\alpha\cos\beta = \sin(\alpha + \beta) + \sin(\alpha - \beta)$.

(27) Verify the identity $2\sin\alpha\sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$.

(28) Verify the identity $\cos 2\theta = 2\sin(\pi/4 + \theta)\sin(\pi/4 - \theta)$

(29) Verify the identity $\dfrac{\sin 2A}{2} = \dfrac{\tan A}{1 + \tan^2 A}$.

(30) Verify the identity $\cot(x/2) = \dfrac{1 + \cos x}{\sin x}$.

(31) Verify the identity $\sin 2B(\cot B + \tan B) = 2$.

(32) Verify the identity $\dfrac{1 - \tan^2\theta}{1 + \tan^2\theta} = \cos 2\theta$.

(33) Verify the identity $1 + \cos 2A = \dfrac{2}{1 + \tan^2 A}$.

(34) Verify the identity $\tan 2x \tan x + 2 = \dfrac{\tan 2x}{\tan x}$.

(35) Verify the identity $\csc A \sec A = 2\csc 2A$.

(36) Verify the identity $\cot x = \dfrac{\sin 2x}{1 - \cos 2x}$.

(37) Verify the identity $1 - \sin A = \left(\sin\dfrac{A}{2} - \cos\dfrac{A}{2}\right)^2$.

(38) Verify the identity $\cos^4 A = \dfrac{2\cos 2A + \cos^2 2A + 1}{4}$.

(39) Verify the identity $\dfrac{\sin A + \sin B}{\sin A - \cos A} = \dfrac{\tan\left(\dfrac{A+B}{2}\right)}{\tan\left(\dfrac{A-B}{2}\right)}$.

(40) Verify the identity $\dfrac{\sin\alpha + \sin 3\alpha}{\cos\alpha + \cos 3\alpha} = \tan 2\alpha$.

(41) Verify the identity $\dfrac{\cos 2A}{1 + \sin 2A} = \dfrac{\cot A - 1}{\cot A + 1}$.

(42) Verify the identity $\dfrac{\cos A + \sin A}{\cos A - \sin A} = \dfrac{1 + \sin 2A}{\cos 2A}$.

(43) Verify the identity $\cot\alpha - \cot\beta = \dfrac{\sin(\beta - \alpha)}{\sin\alpha \sin\beta}$.

(44) Verify the identity $\tan\theta \csc\theta \cos\theta = 1$.

(45) Verify the identity $\cos^2\theta = \dfrac{\cot^2\theta}{1 + \cot^2\theta}$.

(46) Verify the identity $\dfrac{1 - \sin A}{1 + \sin A} = (\sec A - \tan A)^2$.

(47) Verify the identity $(\tan A - \cot A)^2 + 4 = \sec^2 A + \csc^2 A$.

(48) Verify the identity $\cos B \cos(A + B) + \sin B \sin(A + B) = \cos B$.

(49) Verify the identity $\dfrac{\tan A - \sin A}{\sec A} = \dfrac{\sin^3 A}{1 + \cos A}$.

(50) Verify the identity $\dfrac{2\tan^2 A}{1 + \tan^2 A} = 1 - \cos 2A$.

(51) Verify the identity $\tan 2A = \tan A + \dfrac{\tan A}{\cos 2A}$.

(52) Verify the identity $\sin 2A = \dfrac{2\tan A}{1 + \tan^2 A}$.

(53) Verify the identity $\dfrac{4\sin A}{1 - \sin^2 A} = \dfrac{1 + \sin A}{1 - \sin A} - \dfrac{1 - \sin A}{1 + \sin A}$.

(54) Verify the identity $\tan A + \sin A = \dfrac{\csc A + \cot A}{\csc A \cot A}$.

5. Inverse function identities

(1) Verify the identity $\cos\left(\tan^{-1} x\right) = \dfrac{1}{\sqrt{1 + x^2}}$.

(2) Verify the identity $\sin\left(\tan^{-1} x\right) = \dfrac{x}{\sqrt{1+x^2}}$.

(3) Verify the identity $\sin\left(\cos^{-1} x\right) = \sqrt{1-x^2}$.

(4) Verify the identity $\tan\left(\cos^{-1} x\right) = \dfrac{\sqrt{1-x^2}}{x}$.

(5) Verify the identity $\cos\left(\sin^{-1} x\right) = \sqrt{1-x^2}$.

(6) Verify the identity $\tan\left(\cot^{-1} x\right) = \dfrac{1}{x}$.

(7) Verify the identity $\cot\left(\cot^{-1} 2\right) = 2$.

(8) Verify the identity $\sin\left(\cot^{-1} x\right) = \dfrac{1}{\sqrt{1+x^2}}$.

(9) Verify the identity $\cos\left(\cot^{-1} x\right) = \dfrac{x}{\sqrt{1+x^2}}$.

(10) Verify the identity $\sin^{-1}(-x) = -\sin^{-1} x$.

(11) Verify the identity $\tan^{-1}(-x) = -\tan^{-1} x$.

(12) Verify the identity $\tan^{-1} x = \cot^{-1}(1/x)$.

(13) Verify the identity $\tan^{-1} x = \sin^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right)$.

(14) Verify the identity $\sin^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right)$.

(15) Verify the identity $\operatorname{arcsinh}\left(\dfrac{x}{\sqrt{1-x^2}}\right) = \operatorname{arctanh} x$.

6. Taylor series

(1) Define the set $\mathbb{Q}[x]$ and give three example of elements of $\mathbb{Q}[x]$.

(2) Define the set $\mathbb{Q}[[x]]$ and give three example of elements of $\mathbb{Q}[[x]]$.

(3) Define the set $\mathbb{Q}(x)$ and give three examples of elements of $\mathbb{Q}(x)$.

(4) Define the set $\mathbb{Q}((x))$ and give three examples of elements of $\mathbb{Q}((x))$.

(5) Let $D : \mathbb{Q}[x] \to \mathbb{Q}[x]$ be a function such that

(D1) If $f, g \in \mathbb{Q}[x]$ then $D(f+g) = D(f) + D(g)$,
(D2) If $c \in \mathbb{Q}$ and $f \in \mathbb{Q}[x]$ then $D(cf) = cD(f)$,
(D3) If $f, g \in \mathbb{Q}[x]$ then $D(fg) = fD(g) + D(f)g$ and
(D4) $D(x) = 1$.

Prove that if $n \in \mathbb{Z}_{>0}$ then $D(x^n) = nx^{n-1}$.

(6) Write out the first ten terms of the series $\sum_{n=0}^{\infty} \frac{x^n}{n+1}$.

(7) Write out the first ten terms of the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

(8) Write out the first ten terms of the series $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$.

(9) Write out the first ten terms of the series $\sum_{n=0}^{\infty} (\text{-}1)^n \frac{x^n}{n+2}$.

(10) Let $D : \mathbb{Q}[[x]] \to \mathbb{Q}[[x]]$ be a function such that

(D1) If $f, g \in \mathbb{Q}[[x]]$ then $D(f+g) = D(f) + D(g)$,
(D2) If $c \in \mathbb{Q}$ and $f \in \mathbb{Q}[[x]]$ then $D(cf) = cD(f)$,
(D3) If $f, g \in \mathbb{Q}[[x]]$ then $D(fg) = fD(g) + D(f)g$ and
(D4) $D(x) = 1$.

Prove that if $f = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 +$ then $c_k = \frac{1}{k!}(D^k f)|_{x=0}$.

(11) Write $\frac{1-x^n}{1-x}$ as an element of $\mathbb{Q}[x]$.

(12) Write e^x as an element of $\mathbb{Q}[[x]]$.

(13) Write $\sin x$ as an element of $\mathbb{Q}[[x]]$.

(14) Write $\sin(1+x)$ as an element of $\mathbb{Q}[[x]]$.

(15) Write $\cos x$ as an element of $\mathbb{Q}[[x]]$.

(16) Write $\dfrac{1}{1-x}$ as an element of $\mathbb{Q}[[x]]$.

(17) Write $(1+x)^7$ as an element of $\mathbb{Q}[[x]]$.

(18) Write $(1+x)^{1/7}$ as an element of $\mathbb{Q}[[x]]$.

(19) Find the power series representation of $\dfrac{1}{1+2x}$.

(20) Find a series representation of $\dfrac{1}{1+x^2}$.

(21) Find a series representation of $\dfrac{x}{1+x}$.

(22) Find the series representation of $\dfrac{1}{(1+x)^2}$.

(23) Find a series representation of $\arctan x$.

(24) Find a series representation of $\log(2+x)$.

(25) Find a series expansion of $\cosh x$.

(26) Find a series expansion of $\dfrac{1}{1-x}$.

(27) Find a series expansion of $\log(1-x)$.

(28) Find a series expansion of $\log(1+x)$.

(29) Find a series expansion of $\dfrac{\log(1+x)}{\log(1-x)}$.

(30) Find a series expansion of $\sinh x$.

(31) Find a series expansion of $\log(1+2x)$.

(32) Find a series expansion of $\sin x$.

(33) Find a series expansion of $\sin(2x)$.

(34) Find a series expansion of $\cos x$.

(35) Find a series expansion of $\dfrac{1}{1+x}$.

(36) Find a series expansion of $\sinh x$.

(37) Find a series expansion of $\log(2x+1)$.

(38) Find a series expansion of $(1+x)^{-2}$.

(39) Find a series expansion of $\sin(\theta^2)$.

(40) Find a series expansion of $x \sin 3x$.

(41) Find a series expansion of $\cos^2 x$.

(42) Find a series expansion of $\dfrac{t}{1+t}$.

(43) Find a series expansion of $\dfrac{z}{e^{2z}}$.

(44) Find a series expansion of $\sin(x^2)$.

(45) Find a series representation of $\int e^{x^3}\,dx$.

(46) Find the power series representation of $\int \frac{\sinh x}{x}\,dx$.

(47) Find an infinite series representation of $\int_{-1}^{1} \frac{\sinh x}{x}\,dx$.

(48) Find a series expansion of $\int \frac{\cosh x - 1}{x^2}.\,dx$.

(49) Find an infinite series representation of $\int_0^1 e^{x^3}\,dx$.

(50) Find a series expansion of $\int_0^t \sin\left(x^2\right)\,dx$

(51) Find the Taylor expansion of e^x at $x = 0$.

(52) Find the Taylor expansion of $\sinh x$ at $x = 0$.

(53) Find the Taylor expansion of $\frac{1}{1-x}$ at $x = 0$.

(54) Find the Taylor expansion of e^x at $x = 2$.

(55) Find the Taylor expansion of $\log x$ at $x = 1$.

(56) Find the Taylor expansion of $\frac{1}{x^2}$ at $x = 1$.

(57) Find the Taylor series for $\sin x$ at the point $a = \frac{1}{4}\pi$.

(58) Find the Taylor series for $\cos x$ at the point $a = \frac{1}{3}\pi$.

(59) Find the Taylor series for $\frac{1}{x}$ at the point $a = 2$.

(60) Find the Taylor series for e^x at the point $a = -3$.

(61) Find a series representation for e^{2x} in powers of $x + 1$.

(62) Find a series representation for $\log x$ in powers of $x - 1$.

(63) Find an alternate expression for the series $\sum_{n=1}^{\infty} nx^n$.

(64) Find an alternate expression for the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

(65) Find an alternate expression for the series $\sum_{n=2}^{\infty} n(n-1)x^n$.

(66) Find the sum of the series $\sum_{n=1}^{\infty} nx^{n-1}$.

(67) Find the sum of the series $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$.

(68) Find the sum of the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$.

(69) Find the sum of the series $\sum_{n=1}^{\infty} \frac{n}{3^{n-1}}$.

(70) Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n2^{n+1}}$.

(71) Find the sum of the series $\sum_{n=1}^{\infty} n(n-1)\left(\frac{1}{4}\right)^n$.

(72) Find an alternate expression for the series $\sum_{n=1}^{\infty} nx^{n-1}$.

(73) Find an alternate expression for the series $\sum_{n=2}^{\infty} n(n-1)x^{n-2}$.

(74) Find an alternate expression for the series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^k x^k}{k}$.

7. References

[Ca] S. Carnie, *620-143 Applied Mathematics, Course materials*, 2006 and 2007.

[Ho] C. Hodgson, *620-194 Mathematics B and 620-211 Mathematics 2 Notes*, Semester 1, 2005.

[Wi] P. Wightwick, *UMEP notes*, 2010.