§1E. Sets

1. *DeMorgan's Laws.* Let A, B, and C be sets. Show that

a) $(A \cup B) \cup C = A \cup (B \cup C)$.

b) $A \cup B = B \cup A$.

c) $A \cup \emptyset = A$.

d) $(A \cap B) \cap C = A \cap (B \cap C)$.

e) $A \cap B = B \cap A$.

f) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

§2E. Functions

1. Let S, T, and U be sets and let $f: S \to T$ and $g: T \to U$ be functions. Show that
a) If f and g are injective then $g \circ f$ is injective.
b) If f and g are surjective then $g \circ f$ is surjective.
c) If f and g are bijective then $g \circ f$ is bijective.

2. Let $f: S \to T$ be a function and let $U \subseteq S$. The **image** of U under f is the subset of T given by

$$f(U) = \{f(u) \mid u \in U\}.$$

Let $f: S \to T$ be a function. The **image** of f is the subset of T given by

$$\operatorname{im} f = \{f(s) \mid s \in S\}.$$

Note that $\operatorname{im} f = f(S)$.

Let $f: S \to T$ be a function and let $V \subseteq T$. The **inverse image** of V under f is the subset of S given by

$$f^{-1}(V) = \{s \in S \mid f(s) \in V\}.$$

Let $f: S \to T$ be a function and let $t \in T$. The **fiber** of f over t is the subset of S given by

$$f^{-1}(t) = \{s \in S \mid f(s) = t\}.$$

Note that $f^{-1}(t) = f^{-1}(\{t\})$.

Let $f: S \to T$ be a function. Show that the set $F = \{f^{-1}(t) \mid t \in T\}$ of fibers of the map f is a partition of S.

3. a) Let $f: S \to T$ be a function. Define

$$\begin{array}{rccc} f': & S & \to & \operatorname{im} f \\ & s & \mapsto & f(s). \end{array}$$

Show that the map f' is well defined and surjective.

b) Let $f: S \to T$ be a function and let $F = \{f^{-1}(t) \mid t \in T\}$ be the set of nonempty fibers of f. Define

$$\begin{array}{rccc} \hat{f}: & F & \to & T \\ & f^{-1}(t) & \mapsto & t. \end{array}$$

Show that the map $\hat{f}$ is well defined and injective.

c) Let $f: S \to T$ be a function and let $F = \{f^{-1}(t) \mid t \in T\}$ be the set of nonempty fibers of f. Define

$$\begin{array}{rccc} \hat{f}': & F & \to & \operatorname{im} f \\ & f^{-1}(t) & \mapsto & t. \end{array}$$

Show that the map $\hat{f}'$ is well defined and bijective.

4. Let S be a set. The **power set** of S, 2^S, is the set of all subsets of S.

Let S be a set and let $\{0,1\}^S$ be the set of all functions $f\colon S \to \{0,1\}$. Given a subset $T \subseteq S$ define a function $f_T\colon S \to \{0,1\}$ by

$$f_T(s) = \begin{cases} 0 & \text{if } s \notin T; \\ 1 & \text{if } s \in T. \end{cases}$$

Show that the map

$$\begin{array}{rccc} \psi\colon & 2^S & \to & \{0,1\}^S \\ & T & \mapsto & f_T \end{array}$$

is a bijection.

5. Let $\circ\colon S \times S \to S$ be an associative operation on a set S. An **identity** for $\circ$ is an element $e \in S$ such that $e \circ s = s \circ e = s$, for all $s \in S$.

Let e be an identity for an associative operation $\circ$ on a set S. Let $s \in S$. A **left inverse** for s is an element $t \in S$ such that $t \circ s = e$. A **right inverse** for s is an element $t' \in S$ such that $s \circ t' = e$. An **inverse** for s is an element $s^{-1} \in S$ such that $s \circ s^{-1} = s^{-1} \circ s = e$.

a) Let $\circ$ be an operation on a set S. Show that if S contains an identity for $\circ$ then it is unique.

b) Let e be an identity for an associative operation $\circ$ on a set S. Let $s \in S$. Show that if s has an inverse then it is unique.

6. a) Let S and T be sets and let ι_S and ι_T be the identity maps on S and T respectively. Show that for any function $f\colon S \to T$,

$$\begin{aligned} \iota_T \circ f &= f, \qquad \text{and} \\ f \circ \iota_S &= f. \end{aligned}$$

b) Let $f\colon S \to T$ be a function. Show that if an inverse function to f exists then it is unique. (Hint: The proof is very similar to the proof in Ex. 5b.)

§1P. Sets

1. *DeMorgan's Laws.* Let A, B, and C be sets. Show that

a) $(A \cup B) \cup C = A \cup (B \cup C)$.
b) $A \cup B = B \cup A$.
c) $A \cup \emptyset = A$.
d) $(A \cap B) \cap C = A \cap (B \cap C)$.
e) $A \cap B = B \cap A$.
f) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof.

a) To show: aa) $(A \cup B) \cup C \subseteq A \cup (B \cup C)$.
ab) $A \cup (B \cup C) \subseteq (A \cup B) \cup C$.

aa) Let $x \in (A \cup B) \cup C$.
Then $x \in A \cup B$ or $x \in C$.
So $x \in A$ or $x \in B$ or $x \in C$.
So $x \in A$ or $x \in B \cup C$.
So $x \in A \cup (B \cup C)$.
So $(A \cup B) \cup C \subseteq A \cup (B \cup C)$.

ab) Let $x \in A \cup (B \cup C)$.
Then $x \in A$ or $x \in B \cup C$.
So $x \in A$ or $x \in B$ or $x \in C$.
So $x \in A \cup B$ or $x \in C$.
So $x \in (A \cup B) \cup C$.
So $A \cup (B \cup C) \subseteq (A \cup B) \cup C$.

So $(A \cup B) \cup C = A \cup (B \cup C)$.

b) To show: ba) $A \cup B \subseteq B \cup A$.
bb) $B \cup A \subseteq A \cup B$.

ba) Let $x \in A \cup B$.
Then $x \in A$ or $x \in B$.
So $x \in B$ or $x \in A$.
So $x \in B \cup A$.
So $A \cup B \subseteq B \cup A$.

bb) Let $x \in B \cup A$.
Then $x \in B$ or $x \in A$.
So $x \in A$ or $x \in B$.
So $x \in A \cup B$.
So $B \cup A \subseteq A \cup B$.

So $A \cup B = B \cup A$.

c) To show: ca) $A \cup \emptyset \subseteq A$.
cb) $A \subseteq A \cup \emptyset$.

ca) Proof by contradiction.
Assume $A \cup \emptyset \not\subseteq A$.
Then there exists $x \in A \cup \emptyset$ such that $x \notin A$.
So $x \in \emptyset$.
This is a contradiction to the definition of empty set.
So $A \cup \emptyset \subseteq A$.

cb) Let $x \in A$.
Then $x \in A$ or $x \in \emptyset$.
So $x \in A \cup \emptyset$.
So $A \subseteq A \cup \emptyset$.

So $A \cup \emptyset = A$.

d) To show: da) $(A \cap B) \cap C \subseteq A \cap (B \cap C)$.
db) $A \cap (B \cap C) \subseteq (A \cap B) \cap C$.

da) Let $x \in (A \cap B) \cap C$.
Then $x \in A \cap B$ and $x \in C$.
So $x \in A$ and $x \in B$ and $x \in C$.
So $x \in A$ and $x \in B \cap C$.
So $x \in A \cap (B \cap C)$.
So $(A \cap B) \cap C \subseteq A \cap (B \cap C)$.

db) Let $x \in A \cap (B \cap C)$.
Then $x \in A$ and $x \in B \cap C$.
So $x \in A$ and $x \in B$ and $x \in C$.
So $x \in A \cap B$ and $x \in C$.
So $x \in (A \cap B) \cap C$.
So $A \cap (B \cap C) \subseteq (A \cap B) \cap C$.

So $(A \cap B) \cap C = A \cap (B \cap C)$.

e) To show: ea) $A \cap B \subseteq B \cap A$.
eb) $B \cap A \subseteq A \cap B$.

ea) Let $x \in A \cap B$.
Then $x \in A$ and $x \in B$.
So $x \in B$ and $x \in A$.
So $x \in B \cap A$.
So $A \cap B \subseteq B \cap A$.

eb) Let $x \in B \cap A$.
Then $x \in B$ and $x \in A$.
So $x \in A$ and $x \in B$.
So $x \in A \cap B$.
So $B \cap A \subseteq A \cap B$.

So $A \cap B = B \cap A$.

f) To show: fa) $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.
fb) $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

fa) Let $x \in A \cap (B \cup C)$.
Then $x \in A$ and $x \in B \cup C$.
So $x \in A$ and $x \in B$ or $x \in C$.
So $x \in A$ and $x \in B$, or $x \in A$ and $x \in C$.
So $x \in A \cap B$ or $x \in A \cap C$.
So $x \in (A \cap B) \cup (A \cap C)$.
So $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

fb) Let $x \in (A \cap B) \cup (A \cap C)$.
Then $x \in A \cap B$ or $x \in A \cap C$.
So $x \in A$ and $x \in B$, or $x \in A$ and $x \in C$.
So $x \in A$ and, $x \in B$ or $x \in C$.
So $x \in A$ and $x \in B \cup C$.
So $x \in A \cap (B \cup C)$.
So $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

So $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. $\square$

§2P. Functions

(2.2.3) Proposition. *Let $f: S \to T$ be a function. An inverse function to f exists if and only if f is bijective.*

Proof.

$\Longrightarrow$: Assume $f: S \to T$ has an inverse function $f^{-1}: T \to S$.
To show: a) f is injective.
b) f is surjective.

a) Assume $f(s_1) = f(s_2)$.
To show: $s_1 = s_2$.

$$s_1 = f^{-1}\big(f(s_1)\big) = f^{-1}\big(f(s_2)\big) = s_2.$$

So f is injective.

b) Let $t \in T$.
To show: There exists $s \in S$ such that $f(s) = t$.
Let $s = f^{-1}(t)$.
Then

$$f(s) = f\big(f^{-1}(t)\big) = t.$$

So f is surjective.

So f is bijective.

$\Longleftarrow$: Assume $f: S \to T$ is bijective.
To show: f has an inverse function.
We need to define a function $\varphi: T \to S$.
Let $t \in T$.
Since f is surjective there exists $s \in S$ such that $f(s) = t$.
Define $\varphi(t) = s$.
To show: a) φ is well defined.
b) φ is an inverse function to f.

a) To show: aa) If $t \in T$ then $\varphi(t) \in S$.
ab) If $t_1, t_2 \in T$ and $t_1 = t_2$ then $\varphi(t_1) = \varphi(t_2)$.

aa) It is clear from the definition that $\varphi(t) \in S$.
ab) To show: If $t_1 = t_2$ then $\varphi(t_1) = \varphi(t_2)$.
Assume $t_1, t_2 \in T$ and $t_1 = t_2$.
Let $s_1, s_2 \in S$ such that $f(s_1) = t_1$ and $f(s_2) = t_2$.
Since $t_1 = t_2$, $f(s_1) = f(s_2)$.
Since f is injective this implies that $s_1 = s_2$.
So $\varphi(t_1) = s_1 = s_2 = \varphi(t_2)$.

So φ is well defined.

b) To show: ba) If $s \in S$ then $\varphi\big(f(s)\big) = s$.
bb) If $t \in T$ then $f\big(\varphi(t)\big) = t$.

ba) This is immediate from the definition of φ.
bb) Assume $t \in T$.
Let $s \in S$ be such that $f(s) = t$.
Then

$$f\big(\varphi(t)\big) = f(s) = t.$$

So $\varphi \circ f$ and $f \circ \varphi$ are the identity functions on S and T respectively.

So φ is an inverse function to f. $\square$

(2.2.7) Proposition.

a) *Let S be a set and let $\sim$ be an equivalence relation on S. The set of equivalence classes of the relation $\sim$ is a partition of S.*

b) *Let S be a set and let $\{S_\alpha\}$ be a partition of S. Then the relation defined by*

$$s \sim t, \ \textit{if } s,t \textit{ are in the same } S_\alpha,$$

is an equivalence relation on S.

Proof.

a) To show: aa) If $s \in S$ then s is in some equivalence class.
ab) If $[s] \cap [t] \neq \emptyset$ then $[s] = [t]$.

aa) Let $s \in S$.
Since $s \sim s$, $s \in [s]$.

ab) Assume $[s] \cap [t] \neq \emptyset$.
To show: $[s] = [t]$.
Since $[s] \cap [t] \neq 0$, there is an $r \in [s] \cap [t]$.
So $s \sim r$ and $r \sim t$.
By transitivity, $s \sim t$.
To show: aba) $[s] \subseteq [t]$
abb) $[t] \subseteq [s]$.

aba) Suppose $u \in [s]$.
Then $u \sim s$.
We know $s \sim t$.
So, by transitivity, $u \sim t$.
Therefore $u \in [t]$.
So $[s] \subseteq [t]$.

abb) Suppose $v \in [t]$.
Then $v \sim t$.
We know $t \sim s$.
So, by transitivity, $v \sim s$.
Therefore $v \in [s]$.
So $[t] \subseteq [s]$.

So $[s] = [t]$.
So the equivalence classes form a partition of S.

b) We must show that $\sim$ is an equivalence relation, i.e. that $\sim$ is reflexive, symmetric, and transitive.
To show: ba) $s \sim s$ for all $s \in S$.
bb) If $s \sim t$ then $t \sim s$.
bc) If $s \sim t$ and $t \sim u$ then $s \sim u$.

ba) s and s are in the same S_α so $s \sim s$.

bb) Assume $s \sim t$.
Then s and t are in the same S_α.
So $t \sim s$.

bc) Assume $s \sim t$ and $t \sim u$.
Then s and t are in the same S_α and t and u are in the same S_α.
So s and u are in the same S_α.
So $s \sim u$.

So $\sim$ is an equivalence relation. $\square$

1. *Let S, T, U be sets and let $f\colon S \to T$ and $g\colon T \to U$ be functions.*

a) If f and g are injective then $g \circ f$ is injective.
b) If f and g are surjective then $g \circ f$ is surjective.
c) If f and g are bijective then $g \circ f$ is bijective.

Proof.

a) Assume f and g are injective.
To show: If $s_1, s_2 \in S$ and $(g \circ f)(s_1) = (g \circ f)(s_2)$ then $s_1 = s_2$.
Assume $s_1, s_2 \in S$ and $(g \circ f)(s_1) = (g \circ f)(s_2)$.
Then

$$g\big(f(s_1)\big) = g\big(f(s_2)\big).$$

Thus, since g is injective, $f(s_1) = f(s_2)$.
Thus, since f is injective, $s_1 = s_2$.
So $g \circ f$ is injective.

b) Assume f and g are surjective.
To show: If $u \in U$ then there exists $s \in S$ such that $(g \circ f)(s) = u$.
Assume $u \in U$.
Since g is surjective there exists $t \in T$ such that $g(t) = u$.
Since f is surjective there exists $s \in S$ such that $f(s) = t$.
So

$$\begin{aligned}(g \circ f)(s) &= g\big(f(s)\big)\\ &= g(t)\\ &= u.\end{aligned}$$

So there exists $s \in S$ such that $(g \circ f)(s) = u$.
So $g \circ f$ is surjective.

c) Assume f and g are bijective.
To show: ca) $g \circ f$ is injective.
cb) $g \circ f$ is surjective.

ca) Since f and g are bijective, f and g are injective.
Thus, by a), $g \circ f$ is injective.

cb) Since f and g are bijective, f and g are surjective.
Thus, by b), $g \circ f$ is surjective.

So $g \circ f$ is bijective. $\square$

2. *Let $f: S \to T$ be a function. Then the set $F = \{f^{-1}(t) \mid t \in T\}$ of fibers of the map f is a partition of S.*

Proof.

To show: a) If $s' \in S$ then $s' \in f^{-1}(t)$ for some $t \in T$.
b) If $f^{-1}(t_1) \cap f^{-1}(t_2) \neq \emptyset$ then $f^{-1}(t_1) = f^{-1}(t_2)$.

a) Assume $s' \in S$.
Then $f^{-1}(f(s')) = \{s \in S \mid f(s) = f(s')\}$.
Since $f(s') = f(s')$, $s' \in f^{-1}\big(f(s')\big)$.

b) Assume $f^{-1}(t_1) \cap f^{-1}(t_2) \neq \emptyset$.
Let $s \in f^{-1}(t_1) \cap f^{-1}(t_2)$.
So $f(s) = t_1$ and $f(s) = t_2$.
To show: $f^{-1}(t_1) = f^{-1}(t_2)$.
To show: ba) $f^{-1}(t_1) \subseteq f^{-1}(t_2)$.
bb) $f^{-1}(t_2) \subseteq f^{-1}(t_1)$.

ba) Let $k \in f^{-1}(t_1)$.
Then $f(k) = t_1$
$= f(s)$
$= t_2$.
So $k \in f^{-1}(t_2)$.
So $f^{-1}(t_1) \subseteq f^{-1}(t_2)$.

bb) Let $h \in f^{-1}(t_2)$.
Then $f(k) = t_2$
$= f(s)$
$= t_1$.
So $h \in f^{-1}(t_1)$.
So $f^{-1}(t_2) \subseteq f^{-1}(t_1)$.

So $f^{-1}(t_1) = f^{-1}(t_2)$.

So the set $F = \{f^{-1}(t) \mid t \in T\}$ of fibers of the map f is a partition of S. $\square$

3. *a) Let $f: S \to T$ be a function. Define*

$$f'\colon \begin{array}{ccc} S & \to & \operatorname{im} f \\ s & \mapsto & f(s). \end{array}$$

Then the map f' is well defined and surjective.

b) Let $f: S \to T$ be a function and let $F = \{f^{-1}(t) \mid t \in T\}$ be the set of nonempty fibers of f. Define

$$\hat{f}\colon \begin{array}{ccc} F & \to & T \\ f^{-1}(t) & \mapsto & t. \end{array}$$

Then the map $\hat{f}$ is well defined and injective.

c) Let $f: S \to T$ be a function and let $F = \{f^{-1}(t) \mid t \in T\}$ be the set of nonempty fibers of f. Define

$$\hat{f}'\colon \begin{array}{ccc} F & \to & \operatorname{im} f \\ f^{-1}(t) & \mapsto & t. \end{array}$$

Then the map $\hat{f}'$ is well defined and bijective.

Proof.

a) To show: aa) f' is well defined.
ab) f' is surjective.

aa) To show: aaa) If $s \in S$ then $f'(s) \in \operatorname{im} f$.
aab) If $s_1 = s_2$ then $f'(s_1) = f'(s_2)$.

aaa) Assume $s \in S$.
Then $f'(s) = f(s) \in \operatorname{im} f$ by definition of f' and $\operatorname{im} f$.

aab) Assume $s_1 = s_2$.
Then, by definition of f',

$$f'(s_1) = f(s_1) = f(s_2) = f'(s_2).$$

So f' is well defined.

ab) To show: If $t \in \operatorname{im} f$ then there exists $s \in S$ such that $f'(s) = t$.
Assume $t \in \operatorname{im} f$.
Then $f(s) = t$ for some $s \in S$.
So $f'(s) = f(s) = t$.

So f' is surjective.

b) To show: ba) $\hat{f}$ is well defined.
bb) $\hat{f}$ is injective.

ba) To show: baa) If $f^{-1}(t) \in F$ then $\hat{f}\big(f^{-1}(t)\big) \in T$.
bab) If $f^{-1}(t_1) = f^{-1}(t_2)$ then $\hat{f}\big(f^{-1}(t_1)\big) = \hat{f}\big(f^{-1}(t_2)\big)$.

baa) Assume $f^{-1}(t) \in F$.
Then $\hat{f}\big(f^{-1}(t)\big) = t \in T$, by definition.

bab) Assume $f^{-1}(t_1) = f^{-1}(t_2)$.
Let $s \in f^{-1}(t_1)$.
Then $s \in f^{-1}(t_2)$ also.
So $t_1 = f(s) = t_2$.
Then

$$\hat{f}\big(f^{-1}(t_1)\big) = t_1 = t_2 = \hat{f}\big(f^{-1}(t_2)\big).$$

So $\hat{f}$ is well defined.

bb) To show: If $\hat{f}\big(f^{-1}(t_1)\big) = \hat{f}\big(f^{-1}(t_2)\big)$ then $f^{-1}(t_1) = f^{-1}(t_2)$.
Assume $\hat{f}\big(f^{-1}(t_1)\big) = \hat{f}\big(f^{-1}(t_2)\big)$.
Then $t_1 = t_2$.
To show: $f^{-1}(t_1) = f^{-1}(t_2)$.
This is clearly true since $t_1 = t_2$.
So $\hat{f}$ is injective.

c) By Ex. 2.2.3 b), the function

$$\begin{array}{rccc} \hat{f}\colon & F & \to & T \\ & f^{-1}(t) & \mapsto & t \end{array}$$

is well defined and injective.
By Ex. 2.2.3 a), the function

$$\begin{array}{rccc} \hat{f}'\colon & F & \to & \operatorname{im}\hat{f} \\ & f^{-1}(t) & \mapsto & t \end{array}$$

is well defined and surjective.
To show: ca) $\operatorname{im}\hat{f} = \operatorname{im} f$.
cb) $\hat{f}'$ is injective.

ca) To show: caa) $\operatorname{im}\hat{f} \subseteq \operatorname{im} f$.
cab) $\operatorname{im} f \subseteq \operatorname{im}\hat{f}$.

caa) Assume $t \in \operatorname{im}\hat{f}$.
Then $f^{-1}(t)$ is nonempty.
So there exists $s \in S$ such that $f(s) = t$.
So $t \in \operatorname{im} f$.
So $\operatorname{im}\hat{f} \subseteq \operatorname{im} f$.

cab) Assume $t \in \operatorname{im} f$.
Then there exists $s \in S$ such that $f(s) = t$.
So $f^{-1}(t) \neq \emptyset$.
So $t \in \operatorname{im}\hat{f}$.
So $\operatorname{im} f \subseteq \operatorname{im}\hat{f}$.

So $\operatorname{im}\hat{f} = \operatorname{im} f$.

cb) To show: If $\hat{f}'\big(f^{-1}(t_1)\big) = \hat{f}'\big(f^{-1}t_2)\big)$ then $f^{-1}(t_1) = f^{-1}(t_2)$.
Assume $\hat{f}'\big(f^{-1}(t_1)\big) = \hat{f}'\big(f^{-1}(t_2)\big)$.

So $t_1 = t_2$.
So $f^{-1}(t_1) = f^{-1}(t_2)$.
So $\hat{f}'$ is injective.
So $\hat{f}'$ is well defined and bijective. $\square$

4. *Let S be a set and let $\{0,1\}^S$ be the set of all functions $f: S \to \{0,1\}$. Given a subset $T \subseteq S$ define a function $f_T: S \to \{0,1\}$ by*

$$f_T(s) = \begin{cases} 0 & \text{if } s \notin T; \\ 1 & \text{if } s \in T. \end{cases}$$

Then the map

$$\begin{array}{rccc} \psi: & 2^S & \to & \{0,1\}^S \\ & T & \mapsto & f_T \end{array}$$

is a bijection.

Proof.
To show: a) ψ is well defined.
b) ψ is bijective.

a) To show: aa) If $T \in 2^S$ then $\psi(T) = f_T \in \{0,1\}^S$.
ab) If T_1 and T_2 are subsets of S and $T_1 = T_2$ then $\psi(T_1) = \psi(T_2)$.

aa) It is clear from the definition of f_T that $zz/psi(T) = f_T$ is a function from S to $\{0,1\}$.
ab) Assume T_1 and T_2 are subsets of S and $T_1 = T_2$.
To show: $\psi(T_1) = \psi(T_2)$.
To show: $f_{T_1} = f_{T_2}$.
To show: If $s \in S$ then $f_{T_1}(s) = f_{T_2}(s)$.
Assume $s \in S$.
Case 1: If $s \in T_1$ then, since $T_1 = T_2$, $s \in T_2$.
So

$$f_{T_1}(s) = 1 = f_{T_2}(s).$$

Case 2: If $s \notin T_1$ then, since $T_1 = T_2$, $s \notin T_2$.
So

$$f_{T_1}(s) = 0 = f_{T_2}(s).$$

So $f_{T_1}(s) = f_{T_2}(s)$ for all $s \in S$.
So $f_{T_1} = f_{T_2}$.
So $\psi(T_1) = f_{T_1} = f_{T_2} = \psi(T_2)$.
So ψ is well defined.

b) By virtue of Proposition 2.2.3 we would like to show:
$\psi: 2^S \to \{0,1\}^S$ has an inverse function.
Given a function $f: S \to \{0,1\}$ define

$$T_f = \{s \in S \mid f(s) = 1\}.$$

Define a function $\varphi: \{0,1\}^S \to 2^S$ by

$$\begin{array}{rccc} \varphi: & \{0,1\}^S & \to & 2^S \\ & f & \mapsto & T_f. \end{array}$$

To show: ba) φ is well defined.
bb) φ is an inverse function to ψ.

ba) To show: baa) If $f \in \{0,1\}^S$ then $\varphi(f) = T_f \in 2^S$.
bab) If $f_1, f_2 \in \{0,1\}^S$ and $f_1 = f_2$ then

$$\varphi(f_1) = \varphi(f_2).$$

baa) By definition, $T_f = \{s \in S \mid f(s) = 1\}$ is a subset of S.

bab) Assume $f_1, f_2 \in \{0,1\}^S$ and $f_1 = f_2$.
To show: $\varphi(f_1) = \varphi(f_2)$.
To show: $T_{f_1} = T_{f_2}$.
To show: baba) $T_{f_1} \subseteq T_{f_2}$.
babb) $T_{f_2} \subseteq T_{f_1}$.

baba) Assume $s \in T_{f_1}$.
Then $f_1(s) = 1$.
Since $f_2(s) = f_1(s)$, $f_2(s) = 1$.
Thus $s \in T_{f_2}$.
So $T_{f_1} \subseteq T_{f_2}$.

babb) Assume $s \in T_{f_2}$.
Then $f_2(s) = 1$.
Since $f_1(s) = f_2(s)$, $f_1(s) = 1$.
Thus $s \in T_{f_1}$.
So $T_{f_2} \subseteq T_{f_1}$.

So $T_{f_1} = T_{f_2}$.
So $\varphi(f_1) = \varphi(f_2)$.

So φ is well defined.

bb) To show: bba) If $T \in 2^S$ then $\varphi\big(\psi(T)\big) = T$.
bbb) If $f \in \{0,1\}^S$ then $\psi\big(\varphi(f)\big) = f$.

bba) Assume $T \subseteq S$.
To show: $\varphi\big(\psi(T)\big) = T$.
To show: $T_{f_T} = T$.
To show: bbaa) $T_{f_T} \subseteq T$.
bbab) $T \subseteq T_{f_T}$.

bbaa) Assume $t \in T_{f_T}$.
Then $f_T(t) = 1$.
So $t \in T$.
So $T_{f_T} \subseteq T$.

bbab) Assume $t \in T$.
Then $f_T(t) = 1$.
So $t \in T_{f_T}$.
So $T \subseteq T_{f_T}$.

So $T_{f_T} = T$.
So $\varphi\big(\psi(T)\big) = T$.

bbb) Assume $f \in \{0,1\}^S$.
To show: $\psi\big(\varphi(f)\big) = f$.
By definition, $\psi\big(\varphi(f)\big) = f_{T_f}$.
To show: If $s \in S$ then $f_{T_f}(s) = f(s)$.
Assume $s \in S$.

Case 1: $f(s) = 1$.
Then $s \in T_f$.

So $f_{T_f}(s) = 1$.
So $f_{T_f}(s) = f(s)$.
Case 2: $f(s) = 0$.
Then $s \notin T_f$.
So $f_{T_f}(s) = 0$.
So $f_{T_f}(s) = f(s)$.
So $f_{T_f}(s) = f(s)$.
So $\psi\big(\varphi(f)\big) = f$.
So φ is an inverse function to ψ.
So ψ is bijective. $\square$

5. *a) Let $\circ$ be an operation on a set S. If S contains an identity for $\circ$ then it is unique.*
b) Let e be an identity for an associative operation $\circ$ on a set S. Let $s \in S$. If s has an inverse then it is unique.

Proof.
a) Let e, $e' \in S$ be identities for $\circ$.
Then $e \circ e' = e$, since e' is an identity, and $e \circ e' = e'$, since e is an identity.
So $e = e'$.
b) Assume t, $u \in S$ are both inverses for s.
By associativity of $\circ$, $u = (t \circ s) \circ u = t \circ (s \circ u) = t$. $\square$

6. *a) Let S and T be sets and let ι_S and ι_T be the identity maps on S and T respectively. For any function $f\colon S \to T$,*

$$\iota_T \circ f = f, \qquad \text{and}$$
$$f \circ \iota_S = f.$$

b) Let $f\colon S \to T$ be a function. If an inverse function to f exists then it is unique.

Proof.
a) Assume $f\colon S \to T$ is a function.
To show: aa) $\iota_T \circ f = f$.
ab) $f \circ \iota_S = f$.
To show: aa) If $s \in S$ then $\iota_T(f(s)) = f(s)$.
ab) If $s \in S$ then $f(\iota_S(s)) = f(s)$.
aa) and ab) follow immediately from the definitions of ι_T and ι_S respectively.
b) Assume φ and ψ are both inverse functions to f.
To show: $\varphi = \psi$.
By the definitions if identity functions and inverse functions,

$$\varphi = \varphi \circ (f \circ \psi) = (\varphi \circ f) \circ \psi = \psi.$$

So, if an inverse function to f exists, then it is unique. $\square$