Math 521: Lecture 30

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1 The space $C(X)$

Let X be a topological space. Let

$$\mathcal{M}(X) = \{f\colon X \to \mathbb{C}\}$$

be the algebra of complex valued functions on X.

The $*$ **operation** on $\mathcal{M}(X)$ is the map $*\colon \mathcal{M}(X) \to \mathcal{M}(X)$ given by

$$f^*(x) = \overline{f(x)}, \qquad \text{for } x \in X.$$

Let X be a topological space. Let

$$C(X) = \{f\colon X \to \mathbb{R} \mid f \text{ is continuous and bounded}\}.$$

The **supremum norm** on $C(X)$ is the function $\|\ \|\colon C(X) \to \mathbb{R}$ given by

$$\|f\| = \sup_{x\in X} |f(x)|.$$

Define $d\colon C(X) \times C(X) \to \mathbb{R}_{\geq 0}$ by

$$d(f,g) = \|f-g\|.$$

Theorem 1.1. *$C(X)$ is a complete metric space.*

2 Sequences of functions

Let X be a topological space. Let $\mathcal{M}(X)$ be the algebra of complex valued functions on X.

Let $(f_1, f_2, \ldots)$ be a sequence of functions in $\mathcal{M}(X)$.

The sequence $(f_1, f_2, \ldots)$ **converges pointwise** to $f\colon X \to \mathbb{C}$ if

$$f(x) = \lim_{n\to\infty} f_n(x), \qquad \text{for } x \in X.$$

The sequence $(f_1, f_2, \ldots)$ **converges uniformly** to $f\colon X \to \mathbb{C}$ if it is such that if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq 0}$ with $n \geq N$ then

$$|f_n(x) - f(x)| \leq \varepsilon, \qquad \text{for all } x \in X.$$

Proposition 1. *Let* $(f_1, f_2, \ldots)$ *be a sequence of functions in* $C(X)$. *Then* $(f_1, f_2, \ldots)$ *converges in* $C(X)$ *if and only if* $(f_1, f_2, \ldots)$ *converges uniformly.*

Theorem 2.1. *Let* X *be a metric space and let* $E \subseteq X$. *Let* x *be a limit point of* E. *Let* $(f_1, f_2, \ldots)$ *be a sequence of functions in* $\mathcal{M}(E)$ *and suppose that*

$$\lim_{t\to x} f_n(t) \quad \text{exists for each } n \in \mathbb{Z}_{>0}.$$

Then

$$\lim_{n\to\infty} \lim_{t\to x} f_n(t) = \lim_{t\to x} \lim_{n\to\infty} f_n(t).$$

3 The Stone-Weierstrass theorem

Theorem 3.1. *If* $f\colon [a,b] \to \mathbb{C}$ *is a continuous function then there exists a sequence of polynomials* $(p_1, p_2, \ldots)$ *such that* $(p_1, p_2, \ldots)$ *converges uniformly to* f.

Let X be a metric space and let $E \subseteq X$. Let $\mathcal{A}$ be a subalgebra of $C(E)$.

The algebra $\mathcal{A}$ is **self adjoint** if it is such that if $f \in \mathcal{A}$ then $f^* \in \mathcal{A}$.

The algebra $\mathcal{A}$ **separates points** if it is such that if $x_1, x_2 \in E$ then there exists $f \in \mathcal{A}$ such that $f(x_1) \neq f(x_2)$.

The algebra $\mathcal{A}$ **vanishes at no point** if it is such that if $x \in E$ then there exists $f \in \mathcal{A}$ such that $f(x) \neq 0$.

Theorem 3.2. *Let* X *be a metric space and let* K *be a compact subset of* X. *Let* $\mathcal{A}$ *be a subalgebra of* $C(K)$. *If* $\mathcal{A}$ *is self adjoint, separates points and and vanishes at no point of* K *then* $\mathcal{A}$ *is dense in* $C(K)$.