Math 521: Lecture 11

Arun Ram
University of Wisconsin-Madison
480 Lincoln Drive
Madison, WI 53706
ram@math.wisc.edu

1 Ordered sets

Let S be a set. An **partial order** on S is a relation $\leq$ on S such that

(a) If $x, y, z \in S$ and $x \leq y$ and $y \leq z$ then $x \leq z$,

(b) If $x, y \in S$ and $x \leq y$ and $y \leq x$ then $x = y$.

Let S be a set. An **total order** on S is a partial order $\leq$ on S such that

(c) If $x, y \in S$ then $x \leq y$ or $y \leq x$.

A **partially ordered set** or **poset** is a set S with a partial order $\leq$ on S.

Let S be a poset. A **lower order ideal** of S is a subset E of S such that if $y \in E$, $x \in S$ and $x \leq y$ then $x \in E$.

Let S be a poset and let E be a subset of S. An **upper bound** of E is an element $b \in S$ such that if $y \in E$ then $b \geq y$.

Let S be a poset and let E be a subset of S. An **lower bound** of E is an element $\ell \in S$ such that if $y \in E$ then $\ell \leq y$.

Let S be a poset and let E be a subset of S. The **greatest lower bound** of E is the element $\inf(E) \in S$ such that

(a) $\inf(E)$ is a lower bound of E,

(b) if $\ell \in S$ is a lower bound of E then $\ell \leq \inf(E)$.

Let S be a poset and let E be a subset of S. The **least upper bound** of E is an element $\sup(E) \in S$ such that

(a) $\sup(E)$ is a upper bound of E,

(b) if $b \in S$ is a upper bound of E then $\sup(E) \leq b$.

A **lattice** is an poset S such that every pair of elements $x, y \in S$ has a greatest lower bound and a least upper bound.

Let S be a poset. The **intervals** in S are the sets

$$\begin{aligned}
[a,b] &= \{x \in S \mid a \le x \le b\}, \\
[a,b) &= \{x \in S \mid a \le x < b\}, \\
(a,b] &= \{x \in S \mid a < x \le b\}, \\
(a,b) &= \{x \in S \mid a < x < b\}, \\
[a,\infty) &= \{x \in S \mid a \le x\}, \\
(a,\infty) &= \{x \in S \mid a < x\}, \\
(-\infty,b] &= \{x \in S \mid x \le b\}, \\
(-\infty,b) &= \{x \in S \mid x < b\},
\end{aligned}$$

for $a, b \in S$. The sets $[a,b]$, $a, b \in S$ are **closed intervals** and the sets (a,b), $a, b \in S$ are **open intervals**.

HW: Show that if S is a lattice then the intersection of two intervals is an interval. Give an example to show that this is not necessarily true if S is not a lattice.

A poset S is **left filtered** if every subset E of S has an upper bound.

A poset S is **right filtered** if every subset E of S has an lower bound.

Let S be a poset and let E be a subset of S. A **minimal element** of E is an element $x \in E$ such that if $y \in E$ then $x \le y$.

A poset S is **well ordered** if every subset E of S has a minimal element.

HW: Show that Every well ordered set is totally ordered.

HW: Show that there exist totally ordered sets that are not well ordered.