620-295 Real Analysis with Applications

Assignment 6: Due 5pm on 30 October

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Due 5pm on 30 October in the appropriate assignment box on the ground floor of Richard Berry.

1. Determine the area of a parabola topped slice with left edge at $x = l$, right edge at $x = l + 2\Delta x$, middle at $x = l + \Delta x$, left height $f(l)$, middle height $f(l + \Delta x)$, and right height $f(l + 2\Delta x)$.

2. Assume that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Show that $\lim_{x\to a} f(x) + g(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.

3. Assume that $\lim_{x\to a} f(x)$ exists. Show that $\lim_{x\to a} \exp(f(x)) = \exp\left(\lim_{x\to a} f(x)\right)$.

4. Assume that $f(x) = c_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$. Show that $a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)dx$, $a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\, dx$ and $b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\, dx$.

5. Write a quadratic approximation for $f(x) = x^{1/3}$ near 8 and approximate $9^{1/3}$. Estimate the error and find the smallest interval that you can be sure contains the value.

6. Define the following and give an example of each:
   (a) converges pointwise
   (b) converges uniformly
   (b) Taylor series
   (b) Maclaurin series
   (b) Lagrange's remainder
   (b) Riemann's integral
   (b) Trapezoidal integral
   (b) Simpson's integral

7. Carefully state and prove the mean value theorem.

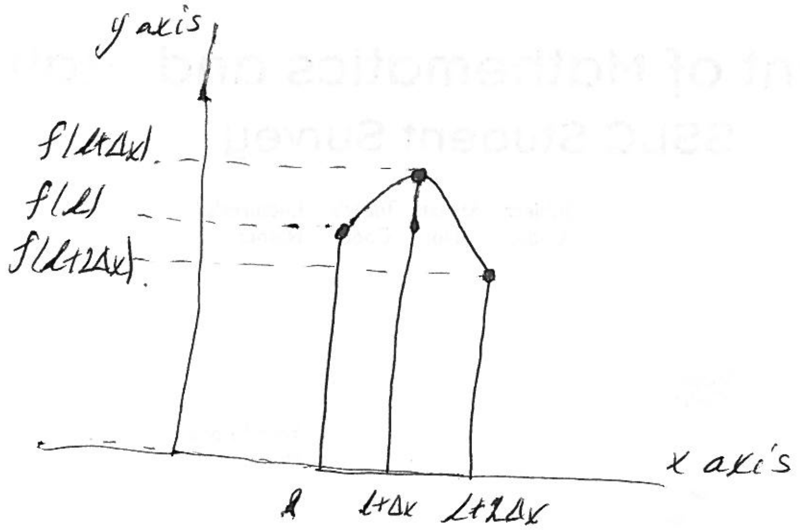
8. (a) Define topological space.
   (b) Define closure of a set.
   (b) Define close point.

(d) Let X be a topological space and let E be a subset of X. Show that the closure of E is equal to the set of close points to E.

9. Assume that $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are functions, if $x \in [a, b]$ then $f(x) \leq g(x)$, and $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ exist. Show that $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.

10. Assume that $f : [a, b] \to \mathbb{R}$ is continuous. Show that there exists $c \in [a, b]$ such that if $x \in [a, b]$ then $f(x) \leq f(c)$ (i.e. f has a maximum at c).

(1) 1.1



Let us find the parabola that goes through the points

$$(l, f(l)), \quad (l+\Delta x, f(l+\Delta x)), \quad (l+2\Delta x), f(l+2\Delta x)).$$

It has equation $y = Ax^2 + Bx + C$ with

$$f(l) = Al^2 + Bl + C$$
$$f(l+\Delta x) = A(l+\Delta x)^2 + B(l+\Delta x) + C$$
$$f(l+2\Delta x) = A(l+2\Delta x)^2 + B(l+2\Delta x) + C$$

The area under the parabola from l to $l+2\Delta x$ is

$$\int_l^{l+2\Delta x} (Ax^2+Bx+C)\,dx = \frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx \Bigg|_{x=l}^{x=l+2\Delta x}$$

$$= \frac{A}{3}(l+2\Delta x)^3 + \frac{B}{2}(l+2\Delta x)^2 + C(l+2\Delta x)$$
$$- \frac{A}{3}l^3 - \frac{B}{2}l^2 - Cl$$

$$= \frac{A}{3}(l^3+3l^2 2\Delta x + 3l\,4(\Delta x)^2 + 8(\Delta x)^3) + \frac{B}{2}(l^2 + 4l\Delta x + 4(\Delta x)^2)$$
$$+ Cl + 2C\Delta x - \frac{A}{3}l^3 - \frac{B}{2}l^2 - Cl$$

$$= 2Al^2\Delta x + 4Al(\Delta x)^2 + \frac{8}{3}A(\Delta x)^3$$
$$+ 2Bl(\Delta x) + 2B(\Delta x)^2$$
$$+ 2C\Delta x$$

$$= \Delta x\left(\begin{array}{c} 2Al^2 + 4Al\,\Delta x + \frac{8}{3}A(\Delta x)^2 \\ + 2Bl + 2B\Delta x + 2C \end{array}\right)$$

$$= \frac{\Delta x}{3}\left(\begin{array}{c} 6Al^2 + 12Al\Delta x + 8A(\Delta x)^2 \\ + 6Bl + 6B\Delta x + 6C \end{array}\right)$$

Since

$$f(l) + 4f(l+\Delta x) + f(l+2\Delta x)$$

$$= Al^2 + Bl + C + 4(A(l+\Delta x)^2 + B(l+\Delta x) + C)$$
$$+ A(l+2\Delta x)^2 + B(l+2\Delta x) + C$$

$$= Al^2 + Bl + C + 4Al^2 + 8Al\Delta x + 4A(\Delta x)^2$$
$$+ 4Bl + 4B\Delta x + 4C + Al^2 + 4Al\Delta x + 4A(\Delta x)^2$$
$$+ Bl + 2B\Delta x + C$$

$$= 6Al^2 + 6Bl + 6C + 12Al\Delta x + 8A(\Delta x)^2$$
$$+ 6B\Delta x$$

we get that the area of the parabola topped slice is

$$\int_l^{l+2\Delta x} (Ax^2 + Bx + C)\,dx = \frac{\Delta x}{3}\left(f(l) + 4f(l+\Delta x) + f(l+2\Delta x)\right)$$

(2.1)

(2) Sum theorem for limits is proved on pages (2) and (3) of lecture 21.

(3) Assume that $\lim_{x \to a} f(x)$ exists.

Show that $\lim_{x \to a} \exp(f(x)) = \exp(\lim_{x \to a} f(x))$

Let $l = \lim_{x \to a} f(x)$.

To show: $\lim_{x \to a} \exp(f(x)) = \exp(l)$

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $|x-a|<\delta$ then $|\exp(f(x)) - \exp(l)| < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$

To show: There exists $\delta \in \mathbb{R}_{>0}$ such that if $|x-a|<\delta$ then $|\exp(f(x)) - \exp(l)| < \varepsilon$.

Let $\delta = \frac{\varepsilon}{e^L}$, where $L = |l|+1$.

To show: If $|x-a|<\delta$ then $|\exp(f(x)) - \exp(l)| < \varepsilon$

Assume $|x-a|<\delta$

To show: $|\exp(f(x)) - \exp(l)| < \varepsilon$.

$$|\exp(f(x)) - \exp(l)|$$
$$\cancel{\le} \left| 1+f(x)+\frac{1}{2!}f(x)^2+\frac{1}{3!}f(x)^3+\cdots - \left(1+l+\frac{1}{2!}l^2+\frac{1}{3!}l^3+\cdots\right)\right|$$
$$= \left| f(x)-l+\frac{1}{2!}(f(x)^2-l^2)+\frac{1}{3!}(f(x)^3-l^3)+\cdots\right|$$

3.2

$$= \left| (f(x)-l)\left(1+\frac{1}{2}(f(x)+l)+\frac{1}{3!}(f(x)^2+f(x)l+l^2)+\cdots\right) \right|$$

$$\leq |f(x)-l| \left| 1+\frac{1}{2}2L+\frac{1}{3!}3L^2+\cdots \right|$$

$$< \delta e^L \leq \varepsilon.$$

So $\lim_{x \to a} \exp(f(x)) = \exp(l)$.

9.1

(4) If $f(x) = c_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$

then

$$f(x) = c_0 + a_1\left(\frac{e^{ix}+e^{-ix}}{2}\right) + b_1\left(\frac{e^{ix}-e^{-ix}}{2i}\right) + a_2\left(\frac{e^{i2x}+e^{-i2x}}{2}\right) + b_2\left(\frac{e^{i2x}-e^{-i2x}}{2i}\right) + \cdots$$

$$= c_0 + \tfrac{1}{2}(a_1 - ib_1)e^{ix} + \tfrac{1}{2}(a_1 + ib_1)e^{-ix} + \tfrac{1}{2}(a_2 - ib_2)e^{i2x} + \tfrac{1}{2}(a_2 + ib_2)e^{-i2x} + \cdots$$

$$= c_0 + \cancel{\tfrac{1}{2}(a_1 - ib_1)}\, c_1 e^{ix} + c_{-1} e^{-ix} + c_2 e^{i2x} + c_{-2} e^{-i2x} + \cdots$$

with

$c_0 = c_0, \quad c_k = \tfrac{1}{2}(a_k - ib_k)$, for $k \in \mathbb{Z}_{>0}$

and $c_{-k} = \tfrac{1}{2}(a_k + ib_k)$, for $k \in \mathbb{Z}_{>0}$.

So $a_k = c_k + c_{-k}$ and $b_k = i(c_k - c_{-k})$.

In class we computed

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-ikx}\,dx .$$

So, if $k \in \mathbb{Z}_{>0}$ then

$$a_k = c_k + c_{-k} = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-ikx}\,dx + \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{ikx}\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} 2f(x)\left(\frac{e^{ikx}+e^{-ikx}}{2}\right) dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx$$

(4.2)

and $$b_k = i(c_k - c_{-k}) = \frac{i}{2\pi}\int_0^{2\pi} f(x)\left(\frac{e^{-ikx} - e^{ikx}}{2i}\right) 2i\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(x)\left(\frac{e^{ikx} - e^{-ikx}}{2i}\right) dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(x) \sin kx \, dx.$$

(5) Write a quadratic approximation for $f(x)=x^{1/3}$ near $x=8$.

Taylor's theorem tells us: There exists $c\in(a,x)$ such that

$$f(x)=f(a)+f'(a)(x-a)+\frac{1}{2!}f''(a)(x-a)^2+\frac{1}{3!}f^{(3)}(c)(x-a)^3.$$

Since $f(x)=x^{1/3}$,

$$f'(x)=\frac{1}{3}x^{-2/3},\quad f''(x)=\frac{1}{3}\left(\frac{-2}{3}\right)x^{-5/3},\quad f^{(3)}(x)=\frac{1}{3}\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)x^{-8/3},$$

and

$$f'(a)=\frac{1}{3}a^{-2/3};\quad f''(a)=-\frac{2}{9}a^{-5/3}\text{ and }f^{(3)}(c)=\frac{10}{27}c^{-8/3}$$

So

$$x^{1/3}=f(x)=a^{1/3}+\frac{1}{3}a^{-2/3}(x-a)-\frac{1}{2}\cdot\frac{2}{9}a^{-5/3}(x-a)^2+\frac{1}{3!}\frac{10}{27}c^{-8/3}(x-a)^3.$$

So the quadratic approximation to $f(x)=x^{1/3}$ near 8 is

$$x^{1/3}\approx 8^{1/3}+\frac{1}{3}8^{-2/3}(x-8)-\frac{1}{9}8^{-5/3}(x-8)^2$$

$$=2+\frac{1}{3}\frac{1}{2^2}(x-8)-\frac{1}{9}\frac{1}{2^5}(x-8)^2$$

$$=2+\frac{1}{12}(x-8)-\frac{1}{9\cdot 32}(x-8)^2,$$

and

$$9^{1/3}\approx 2+\frac{1}{12}(9-8)-\frac{1}{9\cdot 32}(9-8)^2=2+\frac{1}{12}-\frac{1}{288}$$

Then, if $c\in(8,9)$ then $f^{(3)}(c)=\frac{10}{27}c^{-8/3}=\frac{10}{27c^{8/3}}\le\frac{10}{27\cdot 8^{8/3}}=\frac{10}{27\cdot 2^8}$.

so that

$$|\text{Error}|\le\frac{10}{27\cdot 2^8}$$

Thus

$$9^{1/3}\in(A-E,A+E)\text{ where }A=2+\frac{1}{12}-\frac{1}{288}\text{ and }E=\frac{10}{27\cdot 2^8}.$$

(6)
(a) Let X and Y be metric spaces and let $f_n : X \to Y$ be a sequence of functions.

The sequence (f_n) <u>converges pointwise</u> to f if (f_n) satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ and $x \in X$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>0}$ and $n > N$ then $d(f_n(x), f(x)) < \varepsilon$.

(b) The sequence (f_n) <u>converges uniformly</u> to f if (f_n) satisfies:

If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>0}$ and $n > N$ and $x \in X$ then $d(f_n(x), f(x)) < \varepsilon$.

(c) Let $f : [a,b] \to \mathbb{R}$. The <u>Taylor series</u> of degree N at $x = a$ is

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2 + \cdots + \frac{1}{N!} f^{(N)}(a)(x-a)^N + \frac{1}{(N+1)!} f^{(N+1)}(c)(x-a)^{N+1},$$

where $c \in [a,x]$. This expansion exists if $f^{(N)} : [a,b] \to \mathbb{R}$ is continuous and $f^{(N+1)} : (a,b) \to \mathbb{R}$ exists.

(d) The Maclaurin series of f is the Taylor series at $x=0$.

(e) Lagrange's remainder is the term

$$\frac{1}{(N+1)!} f^{(N+1)}(c)(x-a)^{N+1}$$ in the Taylor series of deree N for f at $x=a$.

(f) Let $f:[a,b]\to\mathbb{R}$. The Riemann integral of f is

$$\int_a^b f(x)\,dx = \lim_{\Delta x\to 0} \Delta x\Big(f(a)+f(a+\Delta x)+f(a+2\Delta x)+\cdots + f(b-\Delta x)\Big)$$

(g) The trapezoidal integral of f is

$$\lim_{\Delta x\to 0} \frac{\Delta x}{2}\Big(f(a)+2f(a+\Delta x)+2f(a+2\Delta x)+\cdots+2f(b-\Delta x)+f(b)\Big).$$

(h) The Simpson integral of f is

$$\lim_{\Delta x\to 0}\frac{\Delta x}{3}\Big(f(a)+4f(a+\Delta x)+2f(a+2\Delta x)+4f(a+3\Delta x) + 2f(a+4\Delta x)+4f(a+5\Delta x)+\cdots+4f(b-\Delta x)+f(b)\Big)$$

Still need to put in examples.

(7) The mean value theorem

Let $f : [a,b] \to \mathbb{R}$ be a continuous function and assume that $f' : (a,b) \to \mathbb{R}$ exists. Then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$

Proof

To show: There exists $c \in (a,b)$ such that

$$f(b) = f(a) + f'(c)(b-a)$$

First do the case: $f(a) = f(b)$.

Since $[a,b]$ is compact and connected and $f : [a,b] \to \mathbb{R}$ is continuous,

$f([a,b])$ is compact and connected.

So $f([a,b])$ is a closed and bounded interval.

So $f([a,b]) = [\min, \max]$ for some $\min, \max \in \mathbb{R}$.

~~Thus, there exis~~

So there exists $c \in (a,b)$ such that $f(c) = \max$.

So there exists $\delta \in \mathbb{R}_{>0}$ such that if $|x - c| < \delta$ then $f(x) \le c$.

So $f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \le 0$ and $f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \ge 0$.

So $f'(c) = 0$.

Next do the <u>case</u> $f(a) \neq f(b)$:

Let $g(x) = -\left(\dfrac{f(b)-f(a)}{b-a}\right)(x-a) + f(x)$

so that $g(a) = f(a)$ and $g(b) = f(a)$ and

$$g'(x) = -\left(\frac{f(b)-f(a)}{b-a}\right) + f'(x).$$

Then there exists $c \in (a,b)$ with $g'(c) = 0$.

So $f'(c) - \left(\dfrac{f(b)-f(a)}{b-a}\right) = 0$.

So $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ and $f(b) = f(a) + f'(c)(b-a)$ //.

(8)(a) A <u>topological space</u> is a set X with a collection $\mathcal{T}$ of ~~open~~ subsets of X such that

(a) $\phi \in \mathcal{T}$ and $X \in \mathcal{T}$

(b) If $\mathcal{S} \subseteq \mathcal{T}$ then $\left(\bigcup_{U \in \mathcal{S}} U\right) \in \mathcal{T}$

(c) If $n \in \mathbb{Z}_{>0}$ and $U_1, \ldots, U_n \in \mathcal{T}$ then $U_1 \cap \cdots \cap U_n \in \mathcal{T}$.

(b) Let X be a topological space and let $E \subseteq X$.

The <u>closure</u> of E is the set $\overline{E} \subseteq X$ such that

(a) $\overline{E}$ is closed and $\overline{E} \supseteq E$.

(b) If $K \subseteq X$ and K is closed and $K \supseteq E$ then $K \supseteq \overline{E}$.

(c) Let X be a topological space and let $E \subseteq X$.

A <u>close point</u> to E is an element $x \in X$ such that if N is a neighborhood of x then $N \cap E \neq \phi$

(d) Let X be a topological space and let $E \subseteq X$.

To show: $\overline{E} = \{x \in X \mid x \text{ is a close point of } E\}$

Let $S = \{x \in X \mid x \text{ is a close point of } E\}$.

To show: $\overline{E} = S$.

To show: $(\bar{E})^c = S^c$.

By the theorem that $F^\circ = \{x \in X \mid x \text{ is an interior point of } F\}$

we need to show: (a) $(\bar{E})^c = (E^c)^\circ$

(b) $S^c = \{x \in X \mid x \text{ is an interior point of } E^c\}$

This will give $(\bar{E})^c = (E^c)^\circ = \{x \in X \mid x \text{ is an interior point of } E^c\} = S^c$.

So To show: (a) $(\bar{E})^c = (E^c)^\circ$.

(b) $S^c = \{x \in X \mid x \text{ is an interior point of } E^c\}$.

(a) (1) Since $\bar{E}$ is closed $(\bar{E})^c$ is open.

(2) Since $\bar{E} \supseteq E$, $(\bar{E})^c \subseteq E^c$.

(3) If V is open and $V \subseteq E^c$ then V^c is closed and $V^c \supseteq E$. So $V^c \supseteq \bar{E}$. So $V \subseteq (\bar{E})^c$.

By (1), (2) and (3),

$$(\bar{E})^c = (E^c)^\circ$$

(b) ~~$S^c = \{y \in X \mid y \text{ is not an } \cancel{\text{interior}} \text{ close point of } E\}$~~

Since $S = \{x \in X \mid x \text{ is a close point of } E\}$,

$S^c = \{y \in X \mid y \text{ is not a close point of } E\}$.

So $S^c = \left\{ y \in X \;\middle|\; \begin{array}{l} \text{there exists a neighborhood} \\ N \text{ of } y \text{ such that } N \cap E = \emptyset \end{array} \right\}$

$= \left\{ y \in X \;\middle|\; \begin{array}{l} \text{there exists a neighborhood} \\ N \text{ of } y \text{ such that } N \subseteq E^c \end{array} \right\}$

$= \{ y \in X \mid y \text{ is an interior point of } E^c \}.$

So $(\overline{E})^c = (E^c)^\circ$ and $S^c = \left\{ y \in X \;\middle|\; \begin{array}{l} y \text{ is an interior} \\ \text{point of } E^c \end{array} \right\}$

So $(\overline{E})^c = S^c$.

So $\overline{E} = S$. //

(9) Assume $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ and if $x\in[a,b]$ then $f(x)\le g(x)$.

Assume $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ exist.

The definition of $\int_a^b f(x)dx$ is

$$\int_a^b f(x)dx = \lim_{\Delta x\to 0} \Delta x\left(f(a)+f(a+\Delta x)+\cdots+f(b-\Delta x)\right)$$

$$= \lim_{N\to\infty}\left(\frac{b-a}{N}\right)\left(f(a)+f\left(a+\frac{b-a}{N}\right)+f\left(a+2\left(\frac{b-a}{N}\right)\right)+\cdots\right.$$
$$\left.\cdots+f\left(b-\frac{b-a}{N}\right)\right)$$

To show: $\int_a^b f(x)dx \le \int_a^b g(x)dx$.

$$\int_a^b f(x)dx = \lim_{N\to\infty}\left(\frac{b-a}{N}\right)\left(f(a)+f\left(a+\frac{b-a}{N}\right)+f\left(a+2\left(\frac{b-a}{N}\right)\right)+\cdots\right.$$
$$\left.\cdots+f\left(b-\frac{b-a}{N}\right)\right)$$

$$\le \lim_{N\to\infty}\left(\frac{b-a}{N}\right)\left(g(a)+g\left(a+\frac{b-a}{N}\right)+g\left(a+2\left(\frac{b-a}{N}\right)\right)+\cdots\right.$$
$$\left.\cdots+g\left(b-\frac{b-a}{N}\right)\right)$$

$$= \int_a^b g(x)dx$$

where the inequality follows from one of the theorems for limits:

<u>Theorem</u> If (a_n) and (b_n) are sequences in $\mathbb{R}$ such that ~~and~~ if $n \in \mathbb{Z}_{>0}$ then $a_n \le b_n$ and $\lim_{n\to\infty} a_n$ exists and $\lim_{n\to\infty} b_n$ exists then

$$\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n.$$

<u>Proof</u> Let $A = \lim_{n\to\infty} a_n$ and $B = \lim_{n\to\infty} b_n$.

To show: $A \le B$.

Proof by contradiction.

Assume $A > B$.

Let $\varepsilon \in \mathbb{R}_{>0}$ such that $B < A-\varepsilon < A$.

We know: There exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>0}$ such that $n > N$ then $|a_n - A| < \varepsilon$.

So $a_n > A - \varepsilon$ for $n > N$.

Since $b_n \ge a_n$, then $b_n > A-\varepsilon > B$ for $n > N$.

Let $\varepsilon_1 \in \mathbb{R}_{>0}$ such that $A-\varepsilon > B+\varepsilon_1 > B$.

We know: There exists $N_1 \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>0}$ and $n > N_1$ then $|b_n - B| < \varepsilon_1$

So, if $n > \max(N, N_1)$ then

$b_n > A-\varepsilon > B+\varepsilon_1 > b_n$. This is a contradiction.

So $A \le B$. //

(10) Assume that $f:[a,b]\to\mathbb{R}$ is continuous.

To show: There exists $c\in[a,b]$ such that if $x\in[a,b]$ then $f(x)\le f(c)$.

Since $[a,b]$ is compact and connected and $f:[a,b]\to\mathbb{R}$ is continuous then

$f([a,b])$ is a compact and connected subset of $\mathbb{R}$.

So there exist $l, m\in\mathbb{R}$ so that

$$f([a,b])=[l,m].$$

So there exists $c\in[a,b]$ such that $f(c)=m$.

So, if $x\in[a,b]$ then $f(x)\le m$.

So, if $x\in[a,b]$ then $f(x)\le f(c)$. //

In this proof we used:

(a) If $f:X\to Y$ is continuous and X is connected then $f(X)$ is connected.

(b) If $f:X\to Y$ is continuous and X is compact then $f(X)$ is compact

(c) If $E\subseteq\mathbb{R}$ and E is compact and connected then there exist $l,m\in\mathbb{R}$ such that $E=[l,m]$.

Proof of 1(c)

Assume $E \subseteq \mathbb{R}$ is compact and connected.

Then E is closed and bounded and connected.

~~E~~ Since E is connected E satisfies:

if $x, y \in E$ and $z \in \mathbb{R}$ such that $x < z < y$ then $z \in E$.

Since E is bounded there exist m and l in $\mathbb{R}$ such that if $x \in E$ then $x \leq m$ and $x \geq l$.

Let $m = \sup E$ and $l = \inf E$.

Then $E = [l, m]$.