620-295 Real Analysis with applications, Lect 31, 19.10.2009. ①

Exam preparation

(A) What's on the exam?

(B) Making exams, adjusting length

(C) Marking.

What did we cover?

(1) Numbers $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$

sets, functions, groups, monoids, rings, fields

(2) Orders, ordered fields and absolute value.

(3) limits, sequences and series.

Picard iteration, Newton, Convergence tests.

(4) Continuity and IVP

Topology, Metric spaces, Uniform continuity, Compactness

(5) Taylor's theorem and MVT.

Fourier series.

(6) Derivatives and Integrals.

improper integrals, Trapezoidal, Riemann, Simpson.

(7) Favorite functions

exp, log, trig, hyperbolic, absolute value, rational.

Derivatives

First definition $\frac{d}{dx} : \mathbb{Q}[[x]] \to \mathbb{Q}[[x]]$ such that if $\beta, \gamma \in \mathbb{Q}$ and $f, g \in \mathbb{Q}[[x]]$ then

(a) $\frac{d}{dx}(\beta f + \gamma g) = \beta \frac{df}{dx} + \gamma \frac{dg}{dx}$,

(b) $\frac{d(fg)}{dx} = f \frac{dg}{dx} + \frac{df}{dx} g$,

(c) $\frac{d}{dx}(x) = 1$.

Second definition Let $f : [a,b] \to \mathbb{R}$. Let $c \in [a,b]$.

~~Define~~ The derivative of f at $x = c$ is

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Alternatively,

$$f'(c) = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}.$$

<u>Theorems</u> Let $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ and let $\beta,\gamma\in\mathbb{R}$. Assume that $f'(c)$ and $g'(c)$ exist.

(a) $(\beta f+\gamma g)'(c)=\beta f'(c)+\gamma g'(c)$,

(b) $(fg)'(c)=f'(c)g(c)+f(c)g'(c)$,

(c) Assume $f:[a,b]\to\mathbb{R}$ is given by $f(x)=x$. Then $f'(c)=1$.

(d) If $f'(c)$ exists then f is continuous at $x=c$.

<u>Proof of (d)</u> Assume $f'(c)$ exists.

So $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ exists.

There exists l such that $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=l$.

To show: f is continuous at $x=c$

To show: $\lim_{x\to c} f(x)=f(c)$

To show: If $\varepsilon\in\mathbb{R}_{>0}$ then there exists $\delta\in\mathbb{R}_{>0}$ such that if $|x-c|<\delta$ then $|f(x)-f(c)|<\varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$

We know: There exists $\delta_1 \in \mathbb{R}_{>0}$ such that

if $|x-c| < \delta_1$ then $\left| \frac{f(x)-f(c)}{x-c} - l \right| < \varepsilon/2$

Let $\delta = \min\left(1, \delta_1, \frac{\varepsilon}{2l}\right)$

To show: If $|x-c| < \delta$ then $|f(x)-f(c)| < \varepsilon$.

Assume $|x-c| < \delta$.

To show: $|f(x)-f(c)| < \varepsilon$.

$$|f(x)-f(c)| = \left| \frac{f(x)-f(c)}{x-c} \cdot (x-c) \right|$$

$$= \left| \left(\frac{f(x)-f(c)}{x-c} - l \right)(x-c) + l(x-c) \right|$$

$$\leq \left| \frac{f(x)-f(c)}{x-c} - l \right| |x-c| + l|x-c|$$

$$< \frac{\varepsilon}{2} \cdot \delta + l\delta < \frac{\varepsilon}{2} + \frac{l\varepsilon}{2l} = \varepsilon. \ //$$

Standard derivatives

(a) If $n \in \mathbb{Z}$ then $\frac{d}{dx} x^n = n x^{n-1}$.

(b) If $c \in \mathbb{C}$ then $\frac{d}{dx} x^c = c x^{c-1}$.

(c) $\frac{d}{dx} e^x = e^x$,

(d) $\frac{d}{dx} \log x = \frac{1}{x}$

(e) $\frac{d}{dx} \sin x = \cos x$

(f) $\frac{d}{dx} \arcsin x = \frac{-1}{\sqrt{1-x^2}}$.

The chain rule!

$$\frac{d}{dx}(f \circ g) = \frac{df}{dg}\frac{dg}{dx}.$$